Hyers-Ulam-Rassias Stability of the Inhomogeneous Wave Equation

استقرار حلّ المعادلة الموجية غير المتجانسة بمفهوم هيريس ـ أولام ـ راسيس

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Abstract:

In this paper, we apply the Duhamel's Principle to prove the Hyers-Ulam-Rassias stability for one-dimensional inhomogeneous wave equation on an infinite homogeneous string with zero initial conditions. We have also established the Hyers-Ulam-Rassias stability of nonzero initial value problem of the inhomogeneous wave equation for an infinite string. Some illustrative examples are given.

Keywords: Hyers-Ulam-Rassias Sability, Wave Equation, Duhamel's Principle.

ملخص:

في هذا البحث، استخدم الباحث مبدأ ديوهامل لإثبات استقرار بمعنى هيريس–أولام–راسيس لحلّ المعادلة الموجية غير المتجانسة عبر وتر متجانس ولا نهائي، عندما تكون الشروط الابتدائية صفرية. ولقد أثبت أيضًا الاستقرار للمعادلة الموجية غير المتجانسة عبر وتر متجانس ولا نهائي، عندما تكون الشروط الابتدائية غير صفرية. وجرى دعم النتائج ببعض الأمثلة التوضيحية.

الكلمات المفتاحية: الاستقرار بمفهوم هيريس–أولام–راسيس، المعادلة الموجية، مبدأ ديوهامل.

1- Introduction and Preliminaries

The study of stability problems for various functional equations originated from a famous talk given by Ulam in 1940. In the talk, Ulam discussed a problem concerning the stability of homomorphisms. A significant breakthrough came in 1941, when Hyers [1] gave a partial solution to Ulam's problem. After that and during the last two decades, a great number of papers have been extensively published concerning the various generalizations of Hyers result .(see [2-10]).

Alsina and Ger [11] were the first mathematicians who investigated the Hyers-Ulam stability of the differential equation $g' = g$. They proved that if a differentiable function $y : I \to R$ satisfies $|y' - y| \le \varepsilon,\ \varepsilon > 0,$ for all $t \in I,$ then there exists a differentiable function $g : I \to R$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|g - y| \le 3\varepsilon,$ for all $t \in I.$ This result of Alsina and Ger has been generalized by Takahasi et al [12] to the case of the complex Banach space valued differential equation $y' = \lambda y.$

Furthermore, the results of Hyers-Ulam stability of differential equations of first order were also generalized by Miura et al. [13], Jung [14] and Wang et al. [15].

Gordji et al. [16] generalized Jung's result to first order and second order nonlinear partial differential equations. Lungu and Craciun [17] established results on the Ulam-Hyers stability and the generalized Ulam-Hyers-Rassias stability of nonlinear hyperbolic partial differential equations. Jung [18], Choi and Jung [19] had used coordination substitution way and respectively, the method of a kind of dilation invariance to prove the generalized Hyers-Ulam stability of wave equation. E. Biçer [20] applied Laplace transform technique to establish the Hyers-Ulam stability for the wave equation.

In this paper we consider the Hyers-Ulam-Rassias stability of the nonhomogeneous

wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + g(x,t) \quad 0 < t < \infty, \quad -\infty < x < \infty, \qquad (1.1)$$

with zero initial condition

$$u(x,0) = 0, \ \ u_t(x,0) = 0, \qquad (1.2)$$

where $u(x,t) \in C^2[\mathbb{R} \times (0,\infty)].$

Moreover we have proved sufficient conditions for Hyers-Ulam-Rassias stability of the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + g(x,t), \quad 0 < t < \infty, \quad -\infty < x < \infty \qquad (1.3)$$

with nonzero initial condition

$$u(x,0)=\alpha(x),\ u_t(x,0)=\beta(x),\ -\infty<x<\infty \qquad (1.4)$$

where $u(x,t)\in C^2[\mathbb{R}\times(0,\infty)], \alpha(x)\in C^2(\mathbb{R}^1)$, and $\beta(x)\in C^1(\mathbb{R}^1)$.

Definition 1 [21] We will say that the equation (1.1) has the Hyers-Ulam-Rassias (HUR) stability if there exists $K>0$, $\varphi(x,t):\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and for each solution $u(x,t)\in C^2(\mathbb{R}\times(0,\infty))$ of the inequality

$$|\, u_{tt}-a^2u_{xx}-g(x,t)\,|\le\varphi(x,t)$$

with the initial condition (1.2) then there exists a solution $w(x,t)\in C^2(\mathbb{R}\times(0,\infty))$ of the equation (1.1) such that $|\,u(x,t)-u_0(x,t)\,|\le K\varphi(x,t)$, $\forall(x,t)\in\mathbb{R}\times(0,\infty)$, where K is a constant that does not depend on φ nor on $u(x,t)$, and $\varphi(x,t)\in C(\mathbb{R}\times(0,\infty))$.

Definition 2 [21] We will say that the equation (1.3) has the Hyers-Ulam-Rassias (HUR) stability with respect to $\varphi>0$, if there exists $K>0$ such that for each $\varepsilon>0$ and for each solution $u(x,t)\in C^2(\mathbb{R}\times(0,\infty))$ of the inequality

$$|\, u_{tt}-a^2u_{xx}-g(x,t)\,|\quad\le\varphi(x,t)$$

with the initial condition (1.4) then there exists a solution $w(x,t)\in C^2(\mathbb{R}\times(0,\infty))$ of the equation (1.3) such that $|\,u(x,t)-u_0(x,t)\,|\le K\phi(x,t)$, $\forall(x,t)\in\mathbb{R}\times(0,\infty)$, where K is a constant that does not depend on φ nor on $u(x,t)$, and $\phi(x,t)\in C(\mathbb{R}\times(0,\infty))$.

Now, to motivate the Duhamel method for stability of the infinite homogeneous string in the sense of Hyers-Ulam-Rassias we will consider the following related problem

$$\frac{\partial^2 v}{\partial t^2}=a^2\frac{\partial^2 v}{\partial x^2},\quad t>s,\ -\infty<x<\infty \qquad (1.5)$$

with initial condition

$$v(x,s;s)=0,\ v_t(x,s;s)=g(x,s) \qquad (1.6)$$

where $v(x,t)\in C^2[\mathbb{R}\times(0,\infty)]$.

Now, notice that the problem (1.5),(1.6) has initial conditions prescribed at arbitrary time $t=s$, rather than at $t=0$. Thus we can rewrite $v(x,t;s)=w(x,t-s;s)$ where $w(x,t-s;s)$ solves the problem

$$\frac{\partial^2 w}{\partial t^2}=a^2\frac{\partial^2 w}{\partial x^2},\quad t>s,\ -\infty<x<\infty \qquad (1.7)$$

with initial condition

$$w(x,0;s)=0,\ w_t(x,0;s)=\ g(x,s) \qquad (1.8)$$

where $w(x,t)\in C^2[\mathbb{R}\times(0,\infty)]$.

(Duhamel's Principle for the wave equation (1.1), see [22]) If $g(x,t)\in C^1(\mathbb{R})$ in x and $C^0(0,\infty)$ in t, then

$$u(x,t)=\int_0^t v(x,t;s)ds=\int_0^t w(x,t-s;s)ds$$

$$=\frac{1}{2a}\int_0^t\int_{x-a(t-s)}^{x+a(t-s)} g(r,s)drds \qquad (1.9)$$

2. On Hyers-Ulam-Rassias Stability for Inhomogeneous Wave Equation

First consider the HUR stability of the IV problem (1.3),(1.4) of forced vibrations of a homogeneous infinite string with zero initial conditions.

Theorem1.1 If $v(x,t.;s)\in C^2[\mathbb{R}\times(0,\infty)]$, solves the homogeneous problem (1.7),(1.8), $u(x,t)\in C^2[\mathbb{R}\times(0,\infty)]$ and there is a function $\varphi(x,t):\ \mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$, such that

$$\int_0^t\int_0^s\varphi(x,y)dyds\le K\varphi(x,t) \qquad (2.1)$$

then the IV inhomogeneous problem (1.1),(1.2) is stable in the sense of HUR.

Proof. Let $\varphi(x,t) > 0$ and $u(x,t)$ be an approximate solution of the IV problem (1.1),(1.2)

We will show that there exists a function $u_0(x,t) \in C^2[\mathbb{R} \times (0,\infty)]$ satisfying the equation (1.1) and the initial condition (1.2) such that

$$| u(x,t) - u_0(x,t) | \leq K\varphi(x,t)$$

For (1.1) let consider the inequality

$$-\varphi(x,t) \leq \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} - g(x,t) \quad \leq \varphi(x,t) \tag{2.2}$$

Since $u(x,t) \in C^2[\mathbb{R} \times (0,\infty)]$, we integrate (2.2) with respect to t, to obtain

$$-\int_0^t \varphi(x,s)ds \leq u_t(x,t) - u_t(x,0) - \int_0^t g(x,s)ds - a^2\int_0^t u_{xx}(x,s)ds$$
$$\leq \int_0^t \varphi(x,s)ds \tag{2.3}$$

By virtue of (1.9) and the initial condition (1.8), we have

$$-\int_0^t \varphi(x,s)ds \leq u_t(x,t) - \int_0^t w_t(x,0,s)ds - a^2\int_0^t\int_0^s w_{xx}(x,t-s,s)ds$$
$$\leq \int_0^t \varphi(x,s)ds$$

Or, equivalently

$$-\int_0^t \varphi(x,s)ds \leq u_t(x,t) - \int_0^t w_t(x,0;s)ds - \int_0^t\int_0^s w_{tt}(x,t-s;s)ds$$
$$\leq \int_0^t \varphi(x,s)ds \tag{2.4}$$

Since

$$\frac{\partial}{\partial t}\int_0^t\int_0^s w_t(x,t-s;y)dyds = \int_0^t w_t(x,0;s)ds + \int_0^t w_{tt}(x,t-s;s)ds \tag{2.5}$$

Then by integrating the inequality (2.4) and using (2.1) we get

$$-C\varphi(x,t) \leq -\int_0^t\int_0^s \varphi(x,y)dyds \leq u(x,t) - u(x,0) - \int_0^t\int_0^s w_t(x,t-s;y)dyds$$
$$\leq \int_0^t\int_0^s \varphi(x,y)dyds \leq C\varphi(x,t),$$

Use the initial conditions (1.2),(1.8) to obtain

$$-C\varphi(x,t) \leq u(x,t) - \int_0^t w(x,t-s;s)ds \leq C\varphi(x,t), \tag{2.6}$$

Now, we show that

$$u_0(x,t) = \int_0^t w(x,t-s;s)ds \tag{2.7}$$

satisfies the problem(1.1),(1.2). Indeed Since $w(x,t) \in C^2[\mathbb{R} \times (0,\infty)]$, we differentiate twice, successively with respect to t in order to obtain

$$\frac{\partial}{\partial t}u_0(x,t) = w(x,0;s) + \int_0^t w_t(x,t-s;s)ds = \int_0^t w_t(x,t-s;s)ds \tag{2.8}$$

and

$$\frac{\partial u_{tt}(x,t)}{\partial t} = w_t(x,0;t) + \int_0^t w_{tt}(x,t-s;s)ds$$
$$= g(x,t) + a^2 u_{xx}(x,t),$$

this shows that $u(x,t)$ is a solution of (1.1). The equations (2.7),(2.8) yield $u_0(x,0) = 0$ and $u_t(x,0) = 0$, respectively

By D'Alembert formula, the solution of the problem (1.7), (1.8) is given by

$$w(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} g(r,s)ds \tag{2.10}$$

Hence

$$u_0(x,t) = \int_0^t w(x,t-s;s)ds = \frac{1}{2a}\int_0^t \int_{x-a(t-s)}^{x+a(t-s)} g(r,s)drds \tag{2.11}$$

From (2.6) and (2.7) we infer that the IV problem (1.1), (1.2) is stable in the sense of HUR.

To illustrate the obtained results we give the following example.

Example 2.1 Let the following IV problem be given

$$u_{tt} - u_{xx} = 0 \tag{2.12}$$

$$u(x,0) = 0,\ u_t(x,0) = 0 \tag{2.13}$$

To establish the HUR stability let consider the inequality

$$-\varphi(x,t) \le u_{tt} - u_{xx} - x + t \le \varphi(x,t) \tag{2.14}$$

Integrating (2.14) successively twice with respect to t, the last inequality yield

$$-C\varphi(x,t) \le u(x,t) - \int_0^t\int_0^s u_t(x,t-y;y)\,dyds \le C\varphi(x,t), \tag{2.15}$$

or

$$-C\varphi(x,t) \le u(x,t) - \int_0^t\int_0^s (x-y)\,dyds \le C\varphi(x,t), \tag{2.16}$$

One can show that

$$u_0(x,t) = \int_0^t\int_0^s (x-y)\,dyds = \frac{1}{2}t^2x - \frac{1}{6}t^3 \tag{2.17}$$

is a solution of (2.12),(2.13).

Now if we let

$$\varphi(x,t) = e^{x+t}, \tag{2.18}$$

then from (2.16) and (2.18), we get

$$-C\int_0^t\int_0^s e^{x+t}\,dydt \le u(x,t) - \int_0^t\int_0^s (x-y)\,dyds \le C\int_0^t\int_0^s e^{x+t}\,dydt \tag{2.19}$$

Or, equivalently letting $C=1$ and putting the result

$$\int_0^t\int_0^s e^{(x+s)}\,dydt = e^x\left(e^t - t - 1\right) \le e^{x+t},\ \forall t\in(0,\infty)$$

in (2.19), we have

$$-e^{x+t} \le u(x,t) - \frac{1}{2}t^2x - \frac{1}{6}t^3 \le e^{x+t} \tag{2.20}$$

Hence, the IV problem (2.12), (2.13) is stable in the sense of HUR.

Now we will consider the HUR stability of the IV problem (1.1), (1.2) of forced Vibrations of a homogeneous infinite string with nonzero initial conditions. For this purpose, we consider the following related problems for stability of the infinite homogeneous string in the sense of HUR.

$$\frac{\partial^2 v}{\partial t^2} = a^2\frac{\partial^2 v}{\partial x^2},\quad t>s,\ -\infty<x<\infty \tag{2.21}$$

with initial condition

$$v(x,s;s) = 0,\ v_t(x,s;s) = g(x,s) \tag{2.22}$$

where $v(x,t)\in C^2[\mathbb{R}\times(0,\infty)]$.

Now, notice that the problem (2.21),(2.22) has initial conditions given at arbitrary time $t=s$, instead of at $t=0$. Thus we can rewrite $v(x,t;s) = w(x,t-s;s)$ where $w(x,t-s;s)$ solves the problem

$$\frac{\partial^2 w}{\partial t^2} = a^2\frac{\partial^2 w}{\partial x^2},\quad t>s,\ -\infty<x<\infty \tag{2.23}$$

with initial condition

$$w(x,0;s) = 0,\ w_t(x,0;s) = g(x,s) \tag{2.24}$$

where $w(x,t)\in C^2[\mathbb{R}\times(0,\infty)]$. Now, let $\mu(x,t)\in C^2[\mathbb{R}\times(0,\infty)]$ be a solution of IV Problem

$$\frac{\partial^2 \mu}{\partial t^2} = a^2\frac{\partial^2 \mu}{\partial x^2},\quad t>s,\ -\infty<x<\infty \tag{2.25}$$

with initial condition

$$\mu(x,0;s) = \alpha(x),\ \mu_t(x,0;s) = \beta(x) \tag{2.26}$$

If $v(x,t;s)\in C^2[\mathbb{R}\times(0,\infty)]$, solves the homogeneous problem (2.25),(2.26) and $\varphi(x,t):\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$, such that

$$\int_0^t\int_0^s \varphi(x,y)\,dyds \le K\varphi(x,t)$$

then the IV inhomogeneous problem (1.3),(1.4) is stable in the sense of HUR.

Proof. Let $\varphi(x,t) > 0$ and $u(x,t)$ be an approximate solution of the IV problem (1.3),(1.4). We will show that there exists a function $u_0(x,t) \in C^2[\mathbb{R} \times (0,\infty)]$ satisfying the equation (1.3) and the initial condition (1.4) such that

$$| u(x,t) - u_0(x,t) | \le K\varphi(x,t)$$

First we make a substitution $u(x,t) = \mu(x,t) + w(x,t)$, where $\mu(x,t) \in C^2[\mathbb{R} \times (0,\infty)]$ is a solution of IVP (2.25),(2.26).

Consider the following inequality associated with Eq. (1.3)

$$-\varphi(x,t) \le \frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} - g(x,t) \le \varphi(x,t) \qquad (2.27)$$

By integration (2.27) with respect to t, we have

$$-\int_0^t \varphi(x,s)ds \le u_t(x,t) - u_t(x,0) - \int_0^t g(x,t)ds - a^2\int_0^t u_{xx}(x,t)ds$$

$$\le \int_0^t \varphi(x,s)ds \qquad (2.28)$$

By using (2.28) and the substitution

$$u(x,t) = \mu(x,t) + w(x,t), \qquad (2.29)$$

we obtain

$$-\int_0^t \varphi(x,s)ds \le u_t(x,t) - \mu_t(x,0) - \int_0^t w_t(x,0;s)ds - \int_0^t \mu_t(x,s)ds$$

$$-\int_0^t\int_0^s w_{tt}(x,t-y;y)dyds \le \int_0^t \varphi(x,s)ds \qquad (2.30)$$

Now, in view of the following

$$\frac{\partial}{dt}\left[\int_0^t \mu_t(x,s)ds + \int_0^t\int_0^s w_t(x,t-s;y)dyds\right]$$

$$= \mu_t(x,0;s) + \int_0^t \mu_t(x,s)ds + \int_0^t w_t(x,0;s)ds + \int_0^t\int_0^s w_{tt}(x,t-s;s)ds$$

we get

$$-C\varphi(x,t) \le -\int_0^t\int_0^s \varphi(x,y)dyds \le u(x,t) - u(x,0) - \int_0^t \mu_t(x,s)ds$$

$$-\int_0^t\int_0^s w_t(x,t-y;y)dyds \le \int_0^t\int_0^s \varphi(x,y)dyds \le C\varphi(x,t)$$

Using (2.30) and (2.29) it follows that

$$-C\varphi(x,t) \le u(x,t) - \mu(x,0;s) - w_t(x,0;s) - \int_0^t \mu_t(x,s)ds$$

$$-\int_0^t\int_0^s w_t(x,t-s;s)dsdt \le C\varphi(x,t),$$

Now, we show that

$$u_0(x,t) = \int_0^t \mu_t(x,s)ds + \int_0^t w(x,t-s;s)ds =$$

$$= \mu(x,t) + \int_0^t w(x,t-s;s)ds \qquad (2.31)$$

satisfies the problem(1.3),(1.4). Indeed Since $u_0(x,t) \in C^2[\mathbb{R} \times (0,\infty)]$, we differentiate twice (2.31), successively with respect to t to obtain

$$\frac{\partial}{\partial t}\left[\int_0^t w(x,t-s;s)ds + \mu(x,t)\right] = w(x,0;t) + \int_0^t w_t(x,t-s;s)ds + \mu_t(x,t)$$

$$= \int_0^t w_t(x,t-s;s)ds + \mu_t(x,t) \qquad (2.32)$$

and

$$\frac{\partial^2}{\partial t^2}\left[w(x,0;t) + \int_0^t w_t(x,t-s;s)ds + \mu_t(x,t)\right]$$

$$= w_t(x,0;t) + \int_0^t w_{tt}(x,t-s;s)ds + \mu_{tt}(x,t)$$

$$= g(x,t) + w_{tt}(x,t) + \mu_{tt}(x,t) = g(x,t) + a^2 u_{xx}$$

This shows that $u_0(x,t)$ is a solution of (1.3). From (2.32) and the initial condition (2.26) it follows that $u_0(x,0)=\alpha(x)$ and $u_t(x,0)=\beta(x)$, respectively.

By virtue of (2.31) and applying D' Alembert formula to (2.25) and (2.26), the solution of equation (1.3) is given by

$$u_0(x,t) = \frac{1}{2}\left[\alpha(x+at)+\alpha(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at}\beta(s)ds$$

$$+\frac{1}{2a}\int_0^t\int_{x-a(t-s)}^{x+a(t-s)} g(r,s)\,dr\,ds \qquad (2.33)$$

Therefore, the solution (2.33) of the IV problem (1.3),(1.4) is stable in the sense of HUR.

Example 2.2 Let be given

$$u_{tt} - u_{xx} = x - t \qquad (2.34)$$

$$u(x,0) = x,\quad u_t(x,0) = 1-2x \qquad (2.35)$$

Applying the same argument used in Theorems 2.1,2.3 we obtain

$$u_0(x,t) = \int_0^t w(x,t-s;s)ds + \mu(x,s)$$

where

$$\int_0^t w(x,t-s;s)ds = \int_0^t\int_0^s (x-y)\,dy\,ds = \frac{1}{2}t^2x - \frac{1}{6}t^3$$

is the solution equation (2.34) with corresponding zero initial conditions

$$u(x,0) = 0,\quad u_t(x,0) = 0$$

and

$$\mu(x,t) = x + (1-2x)t$$

is a solution of IV Problem

$$\frac{\partial^2\mu}{\partial t^2} = a^2\frac{\partial^2\mu}{\partial x^2},\quad t>0,\quad -\infty<x<\infty$$

with initial condition

$$\mu(x,0;s) = x,\quad \mu_t(x,0;s) = 1-2x$$

Then by (2.30), we have the solution of IV Problem (2.34),(2.35)

$$u_0(x,t) = x + (1-2x)t + \frac{1}{2}t^2x - \frac{1}{6}t^3$$

One can easily verify that initial condition(2.39) satisfies.

Now let $\varphi(x,t) = e^{x+t}$, then from (2.16),with $C=1$, we get

$$-e^{x+t} \le -\int_0^t\int_0^x e^{x+t}dydt \le u(x,t) - x - (1-2x)t - \frac{1}{2}t^2x + \frac{1}{6}t^3$$
$$\le \int_0^t\int_0^x e^{x+t}dydt \le e^{x+t} \qquad (2.36)$$

By differentiating the inequality (2.36) twice with respect to t, we find

$$-e^{x+t} \le u_{tt}(x,t) - (x-t) \le e^{x+t} \qquad (2.37)$$

Therefore, the inequalities (2.36),(2.37) that the IV problem (2.34),(2.35) is stable in the

Therefore, the inequalities (2.36), (2.37) that the IV problem (2.34),(2.35) are stable in the sense of HUR.

Remark It should be noted here that it follows easily that the solutions (2.11), (2.33) are unique.

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