Reliability and Failure Probability Functions of the Consecutive-*k*-out-of-*m*-from-*n*: F System with Multiple Failure Criteria

اقتران موثوقية و اقتران احتمال فشل النظام التتابعي *k*-out-of-*m*-from-*n*: F متعدد معايير الفشل

Dr. Imad Ismail Nashwan
Associate professor/Al-Quds Open University/Palestine
inashwan@qou.edu

د. عماد إسماعيل نشوان
أستاذ مشارك/ جامعة القدس المفتوحة/ فلسطين

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Abstract:

The consecutive-**k**-out-of-**m**-from-*n*: F system with multiple failure criteria consists of n sequentially ordered components (K= (k_1, k_2,..., k_H), m = (m_1, m_2,..., m_H)). The system fails if among any m_1, m_2,..., m_H consecutive components there are at least k_1, k_2,..., k_H components in the failed state. In this paper, the ordinary consecutive-*k*-out-of-*m*-from-*n*: F system played a pivotal role in achieving the reliability and failure probability functions of the consecutive-**k**-out-of-**m**-from-*n*: F linear and circular system with multiple failure criteria. We proved that the failure states of the multiple failure criteria system is a union of all failure state of the consecutive-k_i-out-of-m_i-from-*n*: F system, while the functioning state is an intersection of the functioning states of the consecutive-k_i-out-of-m_i-from-*n*: F system for $i \in \{1,2,....H\}$. The maximum number of failed components of the functioning consecutive **k**-out-of-**m**-from-*n*: F system with multiple failure criteria is computed.

Keywords: Consecutive *k*-out-of-*m*-from-*n*: F system, Reliability function, Failure probability function

ملخص:

يتكون النظام التتابعي k-out-of-m-from-n: F متعدد معايير الفشل من n من المكونات أو الأجزاء، والذي يفشل إذا حدث انه خلال أي من m_1, m_2,..., m_H المكونات المتتابعة يفشل خلالها على الأقل k_1, k_2,..., k_H عدد من المكونات.

في هذا البحث تم استنتاج اقتران الكثافة الاحتمالي للموثوقية والفشل لهذا النظام من خلال استخدام طبيعة ومكونات النظام العادي ذو الشرط الوحيد k-out-of-m-from-n: F، فلقد أثبتنا أن حالات الفشل للنظام التتابعي متعدد معايير الفشل هو فعليا اتحاد لحالات الفشل للنظام التتابعي ذو الشرط الوحيد(k_i-out-of-m_i-from-*n*: F) ، أما حالات العمل للنظام التتابعي متعدد معايير الفشل فهي تقاطع حالات العمل للنظام التتابعي ذو الشرط الوحيد (k_i-out-of-m_i-from-*n*: F) ، خلال هذا كله، تم حساب العدد الأقصى للمكونات أو الأجزاء التي ممكن أن تفشل بحيث يبقى النظام ككل في حالة العمل.

الكلمات المفتاحية: النظام التتبعي ذو الشرط الوحيد، اقتران موثوقية ، احتمال فشل النظام التتبعي.

Notation

L(C): Linear (circular)

i.i.d.: Independent and identically distributed

I_j^i $= \{i, i+1, ..., j\}$ $1 \le i < j \le n$

$P(I_n^1)$: The power set of I_n^1.

$X = \{x_1, x_2, ..., x_j\}$: A subset of I_n^1, such that $x_i < x_h$ for all $1 \le i < h \le j \le n$

f_n^t: The composite function t times, where $f_n(x) = x \bmod_n + 1 : x \in I_n^1$

$d_X = (d_1^X, d_2^X, ..., d_j^X)$: The rotations of the set $X = \{x_1, x_2, ..., x_j\}$, such that $d_i^X \ge 1$ is the minimum integer number such that $f_n^{d_i^X}(x_i) = x_{i+1}$, for i=1,2,…, j-1, and

$f_n^{d_j^X}(x_j) = x_1$, where $n = \sum_{i=1}^{j} d_i^X$.

$M_r^{L(C)} = (k_r - 1)\overline{[n/m_r]} + s_r^{L(C)}$ where

$$s_r^C = \begin{cases} b_r - m_r + k_r - 1 & b_r \ge m_r - k_r + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$s_r^L = \begin{cases} k_r - 1 & b_r \ge k_r - 1 \\ b_r & b_r < k_r - 1 \end{cases},$$ and

$b_r = n \bmod m_r$.

$d_X^t = (d_{j-t+1}^X, ..., d_j^X, d_1^X, ..., d_{j-t}^X)$ $t \in \mathbf{Z}$, where

$d_X^0 = d_X^j, d_X^{j+s} = d_X^s, t \le j$

$\overline{X}$: The complement of the set X.

$|X|$: The cardinality of the set X.

$\equiv, \sim$: Equivalence relations

$i \underset{j}{\oplus} r$ $= (i+r) \bmod j$, unless if $i + r = nj$ then $i \underset{j}{\oplus} r = j$, when $n \in \mathbf{Z}$

$p_i(q_i)$: The reliability (unreliability) of the i^{th} components

$R(X)(F(X)) = p_X = \prod_{i \in \overline{X}} p_i \prod_{j \in X} q_j$, the reliability (unreliability) of the set X.

$\Psi_{L(C)}^{k,m,n}\left(\Theta_{L(C)}^{k,m,n}\right)$: The collection of all failure (functioning) states of the consecutive-k-out-of-m-from-n: F linear (circular) system.

k,m: Vectors representing failure criteria in the system, ($\mathbf{k} = (k_1, k_2, ..., k_H)$, $\mathbf{m} = (m_1, m_2, ..., m_H)$)

$\Psi_{L(C)}^{\mathbf{k},\mathbf{m},n}\left(\Theta_{L(C)}^{\mathbf{k},\mathbf{m},n}\right)$: The collection of all failure (functioning) states of the consecutive-**k**-out-of-**m**-from-n: F linear (circular) system

p_n^s $= p(n,s) = p^{n-s} q^s$

$\overline{[n]}$: The greatest integer number of n.

1. INTRODUCTION

Over time, the requirements of people's life have become very complicated, requiring highly complex and sophisticated systems. Consequently, this urges the engineers to insure that these systems will perform the required functions. In this context, they developed theorems for such systems, and applied available results for all type of systems, including system reliability, optimal system design, component reliability importance, and reliability bounds.

The consecutive-k-out-of-m-from-n: F system model has interested many engineers since 1985. It is a generalization of the famous consecutive-k-out-of-n: F system which had been used in the telecommunication networks, spacecraft relay stations, vacuum systems in accelerators, oil pipeline systems, photographing nuclear accelerators, microwave stations of a telecom network, etc. Kontoleon (1980) was the first person to introduce the system under the name "r-successive-out-of-n:F system", then Chiang & Niu (1981) created the name "consecutive k-out-of-n: system". Bollinger (1982) presented a direct combinatorial method for determining the system failure probability. Shanthikumar (1982) and Derman et al. (1982) provided a recursive algorithm to evaluate the reliability of the system. Bollinger (1986) introduced a simple and easily programmed algorithm for calculating a table of the coefficients for the failure probability polynomials, associated with the system where the components are i.i.d. Eryılmaz (2009) studied the reliability properties of the consecutive k-out-of-n systems when the components are arbitrarily dependent. Chao M. T, Lin G.D. (1984) and Fu & Hu (1987) studied the reliability of the consecutive k-out-of-n: F system using the Markov chain. Lambiris and Papastavridis (1985) and Nashwan (2015) introduced exact formulas for the reliability of the linear and circular system with i.i.d. components. Dăuş and Beiu (2015) computed the lower and upper bound of the system with a large number of components, and Gökdere (2016) provided a simple way for determining the system failure probability.

The consecutive-k-out-of-m-from-n: F system consists of n components. The components are connected linearly or circularly. The system fails if at least k failed components are included in any m consecutive components. Such a system model was applied in many applications, such as radar detection, quality control and inspection procedures. Tong (1985) was the first to mention the system, while Griffith (1986) introduced the system formally. Afterwards, many researchers studied the system's reliability, failure functions, reliability bounds, optimal system design, etc. Sfakianakis et al. (1992) provided explicit algorithms for the reliability of consecutive-k-out-of-m-from-n: F linear and circular system when the components are i.i.d. Papastavruds & Higsiyama et al. (1995), studied a special case when k=2 with unequal component probabilities. Malinowski & Preuss (1995, 1996) evaluated the reliability of the system with independent component which failure probability may be unequal. Habib et al. (2007) used the total probability theorem to evaluate the reliability of a special case of multi-state consecutive *k*-out-of-*r*-from-*n*: G system. Amirian et al. (2019) provided an algorithm for the exact reliability function of the consecutive k-out-of-r-from-n: F system.

Koutras (1993) provided upper & lower bounds for the reliability of a (linear or circular) consecutive-*k*-out-of-*m*-from-*n*: F system with unequal component failure probabilities. Habib et al. (2000) and Radwan et al. (2011) introduced new bounds for the reliability of the consecutive *k*-out-of-*r*-from-*n*: F system.

The linear consecutive-**k**-out-of-**m**-from-*n*: F system with multiple failure criteria consists also of n connected linearly components. $\mathbf{k}$ and $\mathbf{m}$ are failure integer vector, $\mathbf{k}=\{k_r \mid 1\le r\le H\}$ and $\mathbf{m}=\{m_r \mid 1\le r\le H\}$, where $m_r \le m_{r+1} \le n$, and $k_r \le m_r$. The system fails if at least one group of m_r consecutive components exists in which at least k_r components are in a failed state, for any $1\le r\le H$. One can easily demonstrate that, for any r, if $k_r = 1$, then it becomes a series system.

Actually Levitin (2004) generalized the linear consecutive-*k*-out-of-*r*-from-*n*: F system to the case of multiple failure criteria, and evaluated only the reliability of the system. He used an extended universal moment generating function, and introduced motivated examples as applications, such as the radar system, combat system and the heating system as shown in figure 1.

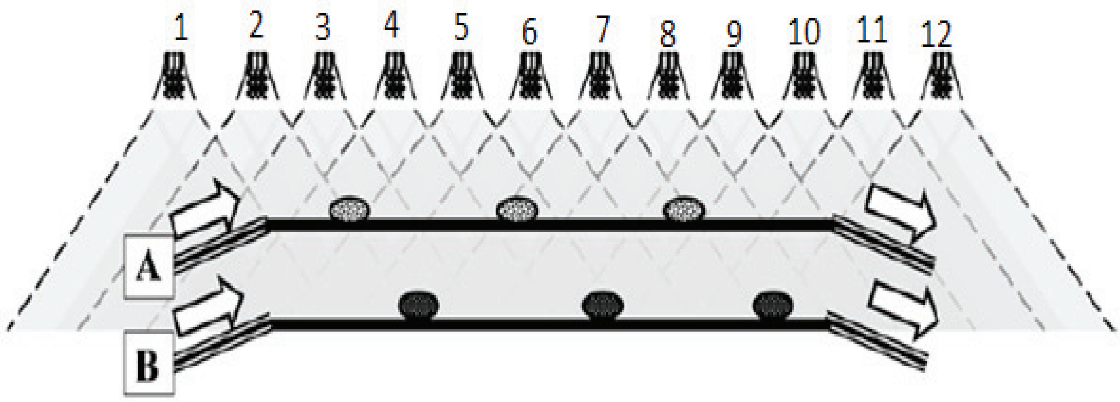

Fig. 1:
The heating system (The linear consecutive-(2,3)-out-of-(3,5)-from-12: F system).

The system consists of 12 heaters, which should provide a certain temperature along the 2 heating lines A and B. The temperature through the two lines at any point is determined by the cumulative effects of the 3 and 5 adjacent heaters, respectively. The heaters cannot provide a certain temperature, if at least 2 out of 3 consecutive heaters, or at least 3 out of 5 consecutive heaters are in the failure state, i.e. the whole system is in failure condition.

In this paper, we developed the classification technique of Nashwan (2018) for the ordinary consecutive-*k*-out-of-*m*-from-*n*: F system (one failure criteria) to compute the exact reliability and failure probability functions of the consecutive-*k*-out-of-*m*-from-*n*: F system with multiple failure criteria. We also developed some conditions to determine the failure and the working states of the system.

In the following section, we study the failure and the functioning states of the circular consecutive-**k**-out-of-**m**-from-*n*: F system with multiple failure criteria using the simple one failure criteria system properties (the circular consecutive-*k*-out-of-*m*-from-*n*: F system). This in turn paved the way to classify them again within the linear type in the third section. Moreover, we computed the maximum possible number of failure components whenever the system is in the functioning state. Finally, an algorithm to find the reliability and failure probability functions of the linear and circular consecutive-**k**-out-of-**m**-from-*n*: F system with multiple failure criteria is obtained. Through all, the system and the components are satisfied by the following:

- The state of the component and the system are either "functioning" or "failed".
- All the components are mutually statistically independent.

2. The circular consecutive-*k*-out-of-*m*-from-*n*: F system with multiple failure criteria

Consider the components indices of the circular consecutive-**k**-out-of-**m**-from-*n*: F system with multiple failure criteria are denoted by $\mathbb{I}_n^1$, $P(\mathbb{I}_n^1)$ is the failure space of the components indices. The system is represented by the set $X=\{x_1,x_2,...,x_j\}\in P(\mathbb{I}_n^1)$, which consists of all the indices of the failed components.

Fix $r\in\mathbb{I}_H^1$, then *X* is a failure state of the system, if there is a m_r consecutive components (whether in the functioning or in the failure state), and among them k_r indices included in *X*, i.e. there is a k_r failed components from *X* among any m_r consecutive components. Moreover, if $Y\in P(\mathbb{I}_n^1)$ such that $X\subseteq Y$, then *Y* is also a failure state. Actually, *X* is a failure state of the simple one criteria circular consecutive-k_r-out-of-m_r-from-*n*: F system.

Conversely, if *X* is a functioning state of the circular consecutive-**k**-out-of-**m**-from-*n*: F system with multiple failure criteria, if it does not hold any failure criteria of the failure vector , i.e. for all $r\in\mathbb{I}_H^1$ *X* is a functioning state in the simple one criteria circular consecutive-k_r-out-of-m_r-from-*n*: F system. In this context, we claim the following:

Claim: $\Psi_C^{\mathbf{k},\mathbf{m},n}=\bigcup_{r=1}^{H}\Psi_C^{k_r,m_r,n}$, and $\Theta_C^{\mathbf{k},\mathbf{m},n}=\bigcap_{r=1}^{H}\Theta_C^{k_r,m_r,n}$.

Proof: If $X\in\Psi_C^{\mathbf{k},\mathbf{m},n}$, then *X* hold at least one failure criteria of the failure vectors, i.e. there exists $r\in\mathbb{I}_H^1$ such that *X* contains at least k_r (indices) failed components among m_r consecutive components, which implies that, *X* is a failure state of the simple circular consecutive-k_r-out-of-m_r-from-n: F system, i.e. $X\in\Psi_C^{k_r,m_r,n}$, which means that $X\in\bigcup_{r=1}^{H}\Psi_C^{k_r,m_r,n}$. Conversely is trivial.

For the functioning states, if $X\in\Theta_C^{\mathbf{k},\mathbf{m},n}$, then it does not hold any criteria of the failure vectors, i.e. for all $r\in\mathbb{I}_H^1$, $X\in\Theta_C^{k_r,m_r,n}$ which implies that $X\in\bigcap_{r=1}^{H}\Theta_C^{k_r,m_r,n}$. Conversely is trivial.

Again, fix any $r\in\mathbb{I}_H^1$, Nashwan (2018) partition $P(\mathbb{I}_n^1)$ of the consecutive-k_r-out-of-m_r-from-*n*: F linear (circular) system into finite pairwise disjoints classes on the form $[X]=\{f_n^{\alpha}(X):\alpha\in\mathbf{Z}\}$, where $f_n:\mathbb{I}_n^1\to\mathbb{I}_n^1$ is a bijection function, such that $f_n(x)=(x\bmod n)+1$ for any $x\in\mathbb{I}_n^1$. He explained that, for any two states $X,Y\in P(\mathbb{I}_n^1)$, $[X]=[Y]$ if there exists $t=1,2,...,j$ such that $d_Y=d_X^t$. Moreover, he classified $P(\mathbb{I}_n^1)$ into two sub collections, $\Psi_{L(C)}^{k_r,m_r,n}$ and $\Theta_{L(C)}^{k_r,m_r,n}$, and computed $M_r^{L(C)}$, the maximum possible failed components, when the consecutive-k_r-out-of-m_r-from-*n*: F linear (circular) system is in the functioning state. For example, in the consecutive-3-from-4-out-of-9: F circular system, the set $X=\{1,2,4,5\}$, for simply $X=1245$, means that, the only failed components are the components with the indices 1,2,4, and 5. The class $[1245]=\{1245,2356,....,1349\}\in\Psi_C^{3,4,9}$, and $[1245]=[2389]$, since $d_{1245}=(1,2,1,5)=d_{2389}^2$, while $M^C=(3-1)\overline{\lceil 9/4\rceil}+0=2\times 2=4$. Moreover, $X=1245\in\Psi_C^{(3,k_2),(4,m_2),9}$ for any integer numbers k_2,m_2, where $k_2\le m_2\le 9$. The next lemma adds more details on the failure and the functioning states of the system.

Lemma 2.1: If the circular consecutive-**k**-out-of-**m**-from-*n*: F system with multiple failure criteria is represented by $X=\{x_1,....,x_j\}\in P(\mathsf{I}_n^1)$ such that $j\geq k=\min_{1\leq r\leq H}\{k_r\}$, define $S_C^X(m_r)=\left\{S_i^X=\sum_{t=0}^{k_r-2}d_{i\oplus_j t}^X : i\in \mathsf{I}_j^1\right\}$, then X is a failed state, if there exists $(i,r)\in \mathsf{I}_j^1\times \mathsf{I}_H^1$ such that $S_i^X\prec m_r$,.

Proof: If there exists $(i,r)\in \mathsf{I}_j^1\times \mathsf{I}_H^1$, such that $S_i^X=\sum_{t=0}^{k_r-2}d_{i\oplus_j t}^X\prec m_r$; hence the total steps on the circle to walk through the k_r distinct failed components $\{x_i,x_{i\oplus_j 1},....,x_{i\oplus_j k_r-1}\}\subseteq X$ is less than m_r steps, i.e. k_r distinct failed components among m_r consecutive components, hence the system fails.

Lemma 2.2: For any two states $X,Y\in P(\mathsf{I}_n^1)$ represent the circular consecutive-k-out-of-m-from-n: F system with multiple failure criteria, such that $Y\in[X]$,

- If $X\in\Psi_C^{\mathbf{k},\mathbf{m},n}\left(\Theta_C^{\mathbf{k},\mathbf{m},n}\right)$, then $Y\in\Psi_C^{\mathbf{k},\mathbf{m},n}\left(\Theta_C^{\mathbf{k},\mathbf{m},n}\right)$.
- $R(Y)=F(Y)=p_{f_n^\alpha(X)}$ for some $\alpha\in\mathbf{Z}$.
- If the components are i.i.d., then $R(Y)=R(X)=p_n^{|X|}$ and $F(Y)=F(X)=p_n^{|X|}$.

Proof:

- If $X\in\Psi_C^{\mathbf{k},\mathbf{m},n}\left(\Theta_C^{\mathbf{k},\mathbf{m},n}\right)$, and $Y\in[X]$, then there exists $t\in \mathsf{I}_j^1$ such that $d_Y=d_X^t$, which implies that $S_C^X(m_r)=S_C^Y(m_r)$, i.e. $Y\in\Psi_C^{\mathbf{k},\mathbf{m},n}\left(\Theta_C^{\mathbf{k},\mathbf{m},n}\right)$.
- If $Y\in[X]$, then there exists $\alpha\in\mathbf{Z}$, such that $Y=f_n^\alpha(X)$, hence , $R(Y)=p_Y=p_{f_n^\alpha(X)}$. Also $F(Y)=p_Y=p_{f_n^\alpha(X)}$
- If the components are i.i.d., then $|X|=|Y|$, apply 2, $R(X)=p_n^{|X|}=p_n^{|Y|}=R(Y)$

Note: The reliability and the failure probability functions of the class $[X]$ are $\mathsf{R}[X]=\sum_{Z\in[X]}R(Z)=\sum_{Z\in[X]}p_Z$, and $\mathsf{F}[X]=\sum_{Z\in[X]}F(Z)=\sum_{Z\in[X]}p_Z$ respectively.

3. The linear consecutive-*k*-out-of-*m*-from-*n*: F system with multiple failure criteria

In this section, the procedure for the system reliability and failure evaluation is based on connecting the 1st and n components in the linear consecutive-**k**-out-of-**m**-from-*n*: F system, and treating the system as a circular type. However, this connection creates more failure states than that in the linear system, i.e. $\Psi_L^{\mathbf{k},\mathbf{m},n}\subseteq\Psi_C^{\mathbf{k},\mathbf{m},n}$ and $\Theta_L^{\mathbf{k},\mathbf{m},n}\supseteq\Theta_C^{\mathbf{k},\mathbf{m},n}$; hence our duty is to separate these extra failures states from $\Psi_C^{\mathbf{k},\mathbf{m},n}$ and add them to $\Theta_C^{\mathbf{k},\mathbf{m},n}$ to compute $\Theta_L^{\mathbf{k},\mathbf{m},n}$.

Lemma 3.1: If the linear consecutive-**k**-out-of-**m**-from -*n*: F system with multiple failure criteria is represented by the set $X=\{x_1,...,x_j\}\in P(\mathsf{I}_n^1)$, such that $j\geq k=\min_{1\leq r\leq H}\{k_r\}$, define $S_L^X(m_r)=\left\{S_i^X=\sum_{t=0}^{k_r-2}d_{i+t}^X : i\in \mathsf{I}_{j-(k_r-1)}^1\right\}$, then X is a failed state if there exists $(i,r)\in \mathsf{I}_{j-(k_r-1)}^1\times \mathsf{I}_H^1$, such that $S_i^X\prec m_r$.

Proof: The proof is the same as in *lemma 2.1* but the condition $i\in \mathsf{I}_{j-(k_r-1)}^1$ is to exclude the effects of the connection between the 1st and the nth components.

For example, the state $\{169\}\in[127]\in\Psi_C^{(2,3),(3,5),9}$, since $S_C^{169}(3)=\{1,3,5\}, S_3^X=1\leq 3$, while $\{169\}\notin\Psi_L^{(2,3),(3,5),9}$, and $S_L^{169}(3)=3,5\geq 3$, and $S_L^{169}(5)=8\geq 5$

Lemma 3.2: Consider the linear (circular) consecutive-**k**-out-of-**m**-from-*n*: F system with multiple failure criteria is in the functioning state, and $M^{L(C)}$ is the maximum number of failed components, then $M^{L(C)} = \min_{1\le r\le H}\left\{M_r^{L(C)}\right\}$.

Proof: Assume that $M^{L(C)} > M_i = \min_{1\le r\le H}\left\{M_r^{L(C)}\right\}$, then WLOG, the consecutive-k_i-out-of-m_i-from-n: F linear (circular) system is in the failure state, which implies that the consecutive-**k**-out-of-**m**-from-*n*: F linear (circular) system is in the failure state, which contradicts the assumption.

4. The proposed algorithm

If *j* is the number of the failed components in the consecutive-**k**-out-of-**m**-from-*n*: F linear and circular system with multiple failure criteria, and $k = \min_{1\le r\le H}\{k_r\}, M^{L(C)} = \min_{1\le r\le H}\left\{M_r^{L(C)}\right\}$, then the failure $\mathsf{F}_j^{L(C)}$ and reliability $\mathsf{R}_j^{L(C)}$ functions are given using the following:

- For j=0,1, …, k-1, all states are in the functioning state, then $\mathsf{R}_j^{L(C)} = \binom{n}{j} p_n^{n-j}$ and $\mathsf{F}_j^{L(C)} = 0$
- Using (lemma 3.2), For j=k, k+1, …, $M^{L(C)}$, find $d_X = \left(d_1^X, d_2^X, ..., d_j^X\right)$.
- Using Nashwan (2018), find the corresponding $X \in P\left(\mathsf{I}_n^1\right)$ and compute $[X]$
- Foreach $r \in \mathsf{I}_H^1$, compute $S_C^X(m_r) = \left\{S_i^X : i \in \mathsf{I}_j^1\right\}$. If there exists $S_i^X \in S_C^X(m_r)$ such that $S_i^X \prec m_r$, then $X \in \Psi_C^{\mathbf{k},\mathbf{m},n}$, otherwise $X \in \Theta_C^{\mathbf{k},\mathbf{m},n}$ (*lemma 2.1*).
- For the linear system,
  - If $X \in \Theta_C^{\mathbf{k},\mathbf{m},n}$ then $[X] \in \Theta_L^{\mathbf{k},\mathbf{m},n}$
  - If $X \in \Psi_C^{\mathbf{k},\mathbf{m},n}$, check $S_L^Y(m_r) = \left\{S_i^Y : i \in \mathsf{I}_{j-(k_r-1)}^1\right\}$ for all $Y \in [X]$.(*lemma 3.1*)
  - Add all Y that does not hold the condition of lemma 3.1 to $\Theta_L^{\mathbf{k},\mathbf{m},n}$.
  - The $\Theta_L^{\mathbf{k},\mathbf{m},n}$ consists of $\Theta_C^{\mathbf{k},\mathbf{m},n}$ and all Y does not hold the condition of *lemma 3.1.* in 5.3.
- Finally, it is obvious that $\mathsf{R}_j^{L(C)}\left(\mathsf{F}_j^{L(C)}\right)$, the summation of the reliability (failure) function of the classes $[X] \in \Theta_{L(C)}^{\mathbf{k},\mathbf{m},n}\left(\Psi_{L(C)}^{\mathbf{k},\mathbf{m},n}\right)$, where $|X| = j$.
- Using (*lemma 3.2*) again, for $j > M^{L(C)}$, all states are failed, hence $\mathsf{F}_j^{L(C)} = \binom{n}{j} p_n^{n-j}$ and $\mathsf{R}_j^{L(C)} = 0$.
- The reliability function of the system is $\mathsf{R}_{L(C)} = \sum_{j=0}^{n} \mathsf{R}_j^{L(C)}$, while the failure function is $\mathsf{F}_{L(C)} = \sum_{j=k}^{n} \mathsf{F}_j^{L(C)}$.

Example 4.1:

The reliability and the failure functions of the (2,3)-out-of-(3,5)-from-9: F linear and circular system

$$M^{L(C)} = 3$$

For j=0, 1 all states are in the functioning states, i.e. $\mathsf{F}_j^{L(C)} = 0, \quad \mathsf{R}_j^{L(C)} = \binom{n}{j} p_n^j$

For j=2

$$d_{\{1,2\}} = (1,8) \Rightarrow S_C(3) = \{1,8\}, S_C(5) = \{9\}$$

$$[12] = \{12,23,34,45,56,67,78,89,\underline{19}\} \in \Psi_C^{(2,3),(3,5),9},$$

$$S_L^{\{19\}}(2) = \{8\} \Rightarrow \{19\} \in \Theta_L^{(2,3),(3,5),9}$$

$$d_{\{1,3\}} = (2,7) \Rightarrow S_C(3) = \{2,7\},$$

$$S_C(5) = \{9\}, W_1^{\{18\}}(2) = W_1^{\{29\}}(2) = 7 \Rightarrow$$

$$[13] = \{13,24,35,46,57,68,79,\underline{18},\underline{29}\} \in \Psi_C^{(2,3),(3,5),9},$$

$$S_L^{\{18\}}(2) = S_L^{\{29\}}(2) = \{8\} \Rightarrow \{18,29\} \in \Theta_L^{(2,3),(3,5),9}$$

$$d_{\{1,4\}} = (3,6) \Rightarrow S_C(3) = \{3,6\}, S_C(5) = \{9\} \Rightarrow$$

$$[14] = \{14,25,36,47,58,69,17,28,39\} \in \Theta_C^{(2,3),(3,5),9}$$

$$d_{\{1,5\}} = (4,5) \Rightarrow S_C(3) = \{4,5\}, S_C(5) = \{9\} \Rightarrow$$

$$[15] = \left\{\begin{matrix}15,26,37,48,59,16,27,\\ 38,49\end{matrix}\right\} \in \Theta_C^{(2,3),(3,5),9}$$

$$\mathsf{F}_2^C = 18p_9^2 \qquad \mathsf{R}_2^C = 18p_9^2$$

$$\mathsf{F}_2^L = 15p_9^2 \qquad \mathsf{R}_2^L = 21p_9^2$$

For j=3

$[123]=\{123,234,345,456,567,678,789,189,129\}\in\Psi_C^{(2,3),(3,5),9}$

$[124]=\{124,235,346,457,568,679,178,289,139\}\in\Psi_C^{(2,3),(3,5),9}$

$[125]=\left\{\begin{matrix}125,236,347,458,569,167,\\278,389,\underline{149}\end{matrix}\right\}\in\Psi_C^{(2,3),(3,5),9}$,

$S_L^{\{149\}}(5)=\{8\}\Rightarrow\{149\}\in\Theta_L^{(2,3),(3,5),9}$

$[126]=\left\{\begin{matrix}126,237,348,459,156,267,\\378,489,\underline{159}\end{matrix}\right\}\in\Psi_C^{(2,3),(3,5),9}$,

$S_L^{\{159\}}(5)=\{8\}\Rightarrow\{159\}\in\Theta_L^{(2,3),(3,5),9}$

$[127]=\left\{\begin{matrix}127,238,349,145,256,367,\\478,589,\underline{169}\end{matrix}\right\}\in\Psi_C^{(2,3),(3,5),9}$,

$S_L^{\{159\}}(5)=\{8\}\Rightarrow\{169\}\in\Theta_L^{(2,3),(3,5),9}$

$[128]=\left\{\begin{matrix}128,239,134,245,356,467,\\578,689,179\end{matrix}\right\}\in\Psi_C^{(2,3),(3,5),9}$

$[135]=\left\{\begin{matrix}135,246,357,468,579,168,\\279,138,249\end{matrix}\right\}\in\Psi_C^{(2,3),(3,5),9}$

$[136]=\left\{\begin{matrix}136,247,358,469,157,268,\\379,\underline{148},\underline{259}\end{matrix}\right\}\in\Psi_C^{(2,3),(3,5),9}$,

$S_L^{\{148\}}(5)=S_L^{\{259\}}(5)=\{7\}\Rightarrow$
$\{148,259\}\in\Theta_L^{(2,3),(3,5),9}$

$[137]=\left\{\begin{matrix}137,248,359,146,257,368,\\479,\underline{158},\underline{269}\end{matrix}\right\}\in\Psi_C^{(2,3),(3,5),9}$,

$S_L^{\{158\}}(5)=S_L^{\{269\}}(5)=\{7\}\Rightarrow\{159,269\}\in\Theta_L^{(2,3),(3,5),9}$

$\mathsf{F}_3^C=81p_9^3 \qquad \mathsf{R}_3^C=3p_9^3$

$\mathsf{F}_3^L=74p_9^3 \qquad \mathsf{R}_3^L=10p_9^3$

For $j\ge 4$ all states are in the failure states, hence $\mathsf{F}_j^{L(C)}=\binom{9}{j}p_9^{9-j}$, $\mathsf{R}_j^{L(C)}=0$, then, the reliability functions of the linear and the circular systems

$$\mathsf{R}_L=\sum_{j=0}^{9}\mathsf{R}_j^C=p_9^0+9p_9^1+21p_9^2+10p_9^3$$
$$\mathsf{R}_C=\sum_{j=0}^{9}\mathsf{R}_j^C=p_9^0+9p_9^1+18p_9^2+3p_9^3,$$

and the failure probability functions of the linear and the circular systems

$$\mathsf{F}_L=\sum_{j=2}^{9}\mathsf{F}_j^C=15p_9^2+74p_9^3+126p_9^4+126p_9^5+81p_9^6+36p_9^7+9p_9^8+p_9^9$$

$$\mathsf{F}_C=\sum_{j=4}^{9}\mathsf{F}_j^C=18p_9^2+81p_9^3+126p_9^4+126p_9^5+81p_9^6+36p_9^7+9p_9^8+p_9^9$$

CONCLUSION

In this paper, we proposed an algorithm to find the reliability and the failure probability functions of the consecutive-**k**-out-**m**-from-*n*: F linear and circular system with multiple failure criteria. In this context, we determined the collections of all failure and the functioning states, where the collection of failure states of the linear type is a sub collection of the circular one. Moreover, we computed the maximum possible number of the failed components in the working consecutive-**k**-out-**m**-from-*n*: F linear and circular systems with multiple failure criteria.

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