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8. Final Remarks

In this paper, we studied the Ergodicity of all the possibilities of non-stationary Markov chains on a finite state space S .

In theorem 7.6, we have the condition $\min_{i,j} p_{ij}^{(n)} = \delta_n \geq \varepsilon > 0$ for infinitely many n 's. This condition is essential for our proof. We can restate this condition in an equivalent form; that is:

There exists a sequence of integers $(r_i)_{i=1}^{\infty}$ such that $A_{r_i+1}A_{r_i+2}\dots A_{r_{i+1}}$ is with non-zero entries and the minimum over all the entries is bounded from below (for infinitely many i 's).

When the state space S is countable, we gave theorem 7.7 with the condition: $\exists j_0 \in S$ such that $\forall i \in S, a_{ij_0} \geq \delta \geq 0$, that is one of the columns of the transition matrix of the limit of the sequence of Markov chains is uniformly bounded. This condition is essential for our proof but most probably it can be weakened.

Non-stationary Markov chains have many applications in the theory of statistical mechanics [9]-[11], [13]. Therefore, we can apply these results in further work, in particular in the theory of Gibbs measures and phase transitions.

The limit of the combination of the elements of a sequence of doubly stochastic Markov chains on an infinite state space can not be Ergodic.

Proof

Assume on the contrary that it is Ergodic. Then

$$\lim_{n \rightarrow \infty} q_{ij}^{(n)} = \pi_j > 0, \forall i \in S \Rightarrow \lim_{n \rightarrow \infty} \sum_{i \in S} q_{ij}^{(n)} = \sum_{i \in S} \pi_j = \infty$$

contradicting the fact that $\sum_{i \in S} q_{ij}^{(n)} = 1$.

Corollary 7.5

In theorems (4) and (6), if the limit of the sequence of the transition matrices A is stable from the first step; that is:

$$\lim_{n \rightarrow \infty} A^n = A, \forall i, j \in S, a_{ij} = a_j.$$

Then

$$\lim_{n \rightarrow \infty} Q_n = \lim_{n \rightarrow \infty} A_1 A_2 \dots A_n = \lim_{n \rightarrow \infty} A^n = \lim_{n \rightarrow \infty} A_n = A.$$

Proof

$$q_{ij}^{(n)} = \sum_{k \in S} q_{ik}^{(n-1)} p_{kj}^{(n)} \leq \sum_{k \in S} q_{ik}^{(n-1)} (a_j + \max_k | \epsilon_{kj}^{(n)} |) = a_j + \max_k | \epsilon_{kj}^{(n)} |.$$

On the other hand,

$$q_{ij}^{(n)} = \sum_{k \in S} q_{ik}^{(n-1)} p_{kj}^{(n)} \geq \sum_{k \in S} q_{ik}^{(n-1)} (a_j - \max_k | \epsilon_{kj}^{(n)} |) = a_j - \max_k | \epsilon_{kj}^{(n)} |.$$

Thus, $| q_{ij}^{(n)} - a_j | \leq \max_k | \epsilon_{kj}^{(n)} | \rightarrow 0$ as $n \rightarrow \infty$.

That is, $\lim_{n \rightarrow \infty} q_{ij}^{(n)} = \lim_{n \rightarrow \infty} p_{ij}^{(n)} = a_j$.

In particular, $p_{ij_0}^{(n)} p_{kj_0}^{(n)} \rightarrow a_{ij_0} a_{kj_0} \geq \delta^2 > 0$. So if we consider the state (j_0, j_0) then:

$$q_{ik}[(X_n, Y_n) = (j_0, j_0) i.o.] = 1$$

and by irreducibility of the combination it is correct for any state (i_0, i_0) .

Thus if T is the first time such that $X_T = Y_T = i_0$, then T is finite with probability 1.

Next,

$$q_{ij}[(X_n, Y_n) = (k, l), T = m] =$$

$$q_{ij}[(X_t, Y_t) \neq (i_0, i_0), t < m, (X_m, Y_m) = (i_0, i_0)] q_{i_0 j_0} [(X_{n-m}, Y_{n-m}) = (k, l)] =$$

$$q_{ij}[T = m] q_{i_0 k}^{(n-m)} q_{i_0 l}^{(n-m)}$$

adding out l gives $q_{ij}[X_n = k, T = m] = q_{ij}[T = m] q_{i_0 k}^{(n-m)}$

adding out k gives $q_{ij}[Y_n = l, T = m] = q_{ij}[T = m] q_{i_0 l}^{(n-m)}$

take $k = l$ and add over $m = 1, 2, \dots, n$

$$\Rightarrow q_{ij}[X_n = k, T \leq n] = q_{ij}[Y_n = k, T \leq n]$$

$$\Rightarrow q_{ij}[X_n = k] \leq q_{ij}[X_n = k, T \leq n] + q_{ij}[T > n] = q_{ij}[Y_n = k, T \leq n] + q_{ij}[T > n]$$

Thus $q_{ij}[X_n = k] \leq q_{ij}[Y_n = k] + q_{ij}[T > n]$. (13)

By similar argument we get,

$$q_{ij}[Y_n = k] \leq q_{ij}[X_n = k] + q_{ij}[T > n]. \quad (14)$$

Inequalities (13) and (14) $\Rightarrow |q_{ik}^{(n)} - q_{jk}^{(n)}| \leq q_{ij}[T > n]$

Since T is finite with probability 1 $\Rightarrow \lim_{n \rightarrow \infty} |q_{ik}^{(n)} - q_{jk}^{(n)}| = 0$

This means that the combination is independent of the initial state.

Next, $q_{ij_0}^{(n)} \geq \delta_n > 0$ since $\delta_n \rightarrow \delta > 0$.

Thus by irreducibility of the combination, $\lim_{n \rightarrow \infty} q_{ik}^{(n)} > 0, \forall k \in S$.

Hence, the limit of the combination exists and Ergodic (by the same argument of the finite case, replacing minimum by infimum and maximum by supremum, the limit is independent of n).

Corollary 7.4

Hence $\lim_{n \rightarrow \infty} q_{ij}^{(n)} = \frac{1}{N}, \forall i, j \in S$.

Theorem 7.7

Assume we have a sequence of Markov chains on a countable state space S , whose limit is an Ergodic Markov chain. Assume that $\exists j_0 \in S$ such that $\forall i \in S, a_{ij_0} \geq \delta \geq 0$, where $[a_{ij}]_{i,j \in S}$ are the entries of the transition matrix of the limit of the sequence, say A . Then the limit of the combination of the elements of this sequence exists and Ergodic.

Proof

Let $(A_n)_{n=1}^{\infty}$ be the transition matrices of this sequence.

Denote the transition matrix of the n-th chain by A_n and its entries by $[p_{ij}^{(n)}]_{i,j \in S}$.

Let $\lim_{n \rightarrow \infty} A_n = A$ and denote its entries by $[a_{ij}]_{i,j \in S}$.

Let $Q_n = A_1 A_2 \dots A_n$. and denote its entries by $[q_{ij}^{(n)}]_{i,j \in S}$.

Denote the entries of $A_{m+1} A_{m+2} \dots A_n$ by $[q_{ij}^{(n-m)}]_{i,j \in S}$.

Assume without loss of generality that A is with non-zero entries and that A_n 's are with non-zero entries (otherwise the same argument of the finite case).

We will use coupling method to prove this theorem [5]¹⁰.

Define a coupled chain on the state space (S, S) with transition probabilities:

$$P[(X_{n+1}, Y_{n+1}) = (k, l) | (X_n, Y_n) = (i, j)] = p^{(n)}(ij, kl) = p_{ik}^{(n)} p_{jl}^{(n)}.$$

Notice that this coupled chain is irreducible (it is with non-zeros).

Since $p_{ik}^{(n)} \rightarrow a_{ik}, p_{jl}^{(n)} \rightarrow a_{jl} \Rightarrow p_{ik}^{(n)} p_{jl}^{(n)} \rightarrow a_{ik} a_{jl}$.

¹⁰ See Chen, 1992, p.6

combination exists and Ergodic. In fact, $\lim_{n \rightarrow \infty} q_{ij}^{(n)} = \frac{1}{N}, \forall i, j \in S$, where N is the cardinality of S .

Proof

By the previous theorem we have Weak Ergodicity, so:

$$\forall j \in S, q_{ij}^{(n)} = q_j^{(n)} + \varepsilon_{ij}^{(n)} \text{ where } \varepsilon_{ij}^{(n)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

$$\text{Next, } \sum_{i \in S} q_{ij}^{(n)} = \sum_{i \in S} (q_j^{(n)} + \varepsilon_{ij}^{(n)}) = 1.$$

$$\text{Thus } Nq_j^{(n)} + \sum_{i \in S} \varepsilon_{ij}^{(n)} = 1 \Rightarrow q_j^{(n)} + \varepsilon_{ij}^{(n)} = 1/N.$$

$$\text{Thus } \forall i, j \in S, q_{ij}^{(n)} = 1/N + \varepsilon_{ij}^{(n)}.$$

$$q_{ij}^{(n+m)} = \sum_{k \in S} q_{ik}^{(n)} b_{kj}^{(m)}, \text{ where}$$

$$[b_{ij}^{(m)}]_{i, j \in S} \text{ are the entries of } [b_{ij}^{(m)}]_{i, j \in S} A_{n+1} A_{n+2} \dots A_{n+m}.$$

$$\Rightarrow q_{ij}^{(n+m)} = \sum_{k \in S} \left(\frac{1}{N} + \varepsilon_{ik}^{(n)} \right) b_{kj}^{(m)}$$

$$\leq \sum_{k \in S} \left(\max_k \left(\frac{1}{N} + \varepsilon_{ik}^{(n)} \right) \right) b_{kj}^{(m)}$$

$$\leq \left(\frac{1}{N} + \max_k |\varepsilon_{ik}^{(n)}| \right) \sum_{k \in S} b_{kj}^{(m)}$$

$$= \frac{1}{N} + \max_k |\varepsilon_{ik}^{(n)}|.$$

On the other hand

$$q_{ij}^{(n+m)} \geq \sum_{k \in S} \left(\min_k \left(\frac{1}{N} + \varepsilon_{ik}^{(n)} \right) \right) b_{kj}^{(m)}$$

$$\geq \left(\frac{1}{N} - \max_k |\varepsilon_{ik}^{(n)}| \right) \sum_{k \in S} b_{kj}^{(m)}$$

$$= \frac{1}{N} - \max_k |\varepsilon_{ik}^{(n)}|.$$

Thus $\lim_{n \rightarrow \infty} |q_{ij}^{(n+m)} - q_{ij}^{(n)}| \leq \max_j |\varepsilon_{ij}^{(n)}| \rightarrow 0$ as $n \rightarrow \infty, \forall j \in S$.

Thus $(R_j^{(2)} - r_j^{(2)}) \leq (1 - N\delta_1)(1 - N\delta_2)$ (10)

Where $R_j^{(2)} = \max_i q_{ij}^{(2)}$ and $r_j^{(2)} = \min_i q_{ij}^{(2)}$.

Next, assume that: $(R_j^{(n-1)} - r_j^{(n-1)}) \leq \prod_{i=1}^{n-1} (1 - N\delta_i)$ (11)

Where $R_j^{(n-1)} = \max_i q_{ij}^{(n-1)}$ and $r_j^{(n-1)} = \min_i q_{ij}^{(n-1)}$.

Now we want to prove that it is correct for n .

$$(q_{uj}^{(n)} - q_{vj}^{(n)}) = \sum_{k \in S} (q_{uk}^{(1)} - q_{vk}^{(1)}) b_{kj}^{(n-1)} \leq \sum^+ (q_{uk}^{(1)} - q_{vk}^{(1)}) B_j^{(n-1)} + \sum^- (q_{uk}^{(1)} - q_{vk}^{(1)}) b_j^{(n-1)} \\ \leq \sum^+ (q_{uk}^{(1)} - q_{vk}^{(1)}) (B_j^{(n-1)} - b_j^{(n-1)}) \leq (1 - N\delta_1) (B_j^{(n-1)} - b_j^{(n-1)})$$

where $[b_{ij}^{(n-1)}]_{i,j \in S}$ are the entries of $A_2 A_3 \dots A_n$ and

$B_j^{(n-1)} = \max_i b_{ij}^{(n-1)}$ and $b_j^{(n-1)} = \min_i b_{ij}^{(n-1)}$.

Thus $(q_{uj}^{(n)} - q_{vj}^{(n)}) \leq (1 - N\delta_1) \prod_{i=2}^n (1 - N\delta_i) = \prod_{i=1}^n (1 - N\delta_i)$.

(9),(10) and (11) $\Rightarrow (R_j^{(n)} - r_j^{(n)}) \leq \prod_{i=1}^n (1 - N\delta_i)$ (12)

Where $R_j^{(n)} = \max_i q_{ij}^{(n)}$ and $r_j^{(n)} = \min_i q_{ij}^{(n)}$.

Hence, by the principle of mathematical induction, (6) is valid for each natural number n .

Next, $\delta_i \geq \varepsilon > 0$ infinitely many $\Rightarrow \prod_{i=1}^n (1 - N\delta_i) \rightarrow 0$ as $n \rightarrow \infty$.

Hence, $\forall j \in S$ and arbitrary $i, k \in S$, we have:

$$\lim_{n \rightarrow \infty} |q_{ij}^{(n)} - q_{kj}^{(n)}| = 0.$$

That is $\lim_n q_{ij}^{(n)}$ is Weak Ergodic; in other words the effect of the initial state wears off.

Corollary 7.3

In the previous theorem, if the sequence of the transition matrices is doubly stochastic; that is each matrix is stochastic and satisfies $\sum_{i \in S} p_{ij}^{(n)} = 1, \forall j \in S$ and $\forall n = 1, 2, \dots$. Then the limit of the

Assume without loss of generality that for all n , A_n is with non-zero entries (otherwise there exists k_0 such that A_n is with non-zero entries for all $n \geq k_0$).

Recall that for any stochastic matrix A with entries $[p_{ij}]_{i,j \in S}$ and minimum over all its entries δ , the following relations are valid:

Denote the summation over $j \in S$ satisfying $p_{uj} \geq p_{vj}$ by \sum^+ and the summation over $j \in S$ satisfying $p_{uj} < p_{vj}$ by \sum^- for arbitrary states u and v in S .

$$\text{Then } \sum^+(p_{uj}-p_{vj}) + \sum^-(p_{uj} - p_{vj}) = 1 - 1 = 0. \dots(7)$$

and since $\sum^+ p_{vj} + \sum^- p_{uj} \geq N\delta$, then :

$$\sum^+(p_{uj}-p_{vj}) = 1 - \sum^- p_{uj} - \sum^+ p_{vj} \leq (1 - N\delta). \dots(8)$$

Next, we will use induction on n to prove that:

$$(\max_i q_{ij}^{(n)} - \min_i q_{ij}^{(n)}) \leq \prod_{i=1}^n (1 - N\delta_i).$$

For $n = 1$:

$$\max_{i,j} p_{ij}^{(1)} \leq (1 - (N - 1)\delta_1) \text{ and}$$

$$\min_{i,j} p_{ij}^{(1)} \leq \delta_1 \Rightarrow (\max_i p_{ij}^{(1)} - \min_i p_{ij}^{(1)}) \leq (1 - N\delta_1)$$

$$\text{and since } Q_1 = P_1 \text{ then } (\max_i p_{ij}^{(1)} - \min_i p_{ij}^{(1)}) \leq (1 - N\delta_1).$$

...(9)

For $n = 2$:

$$(q_{uj}^{(2)} - q_{vj}^{(2)}) = \sum_{k \in S} (q_{uk}^{(1)} - q_{vk}^{(1)}) p_{kj}^{(2)} \leq \sum^+ (q_{uk}^{(1)} - q_{vk}^{(1)}) M_j^{(2)} + \sum^- (q_{uk}^{(1)} - q_{vk}^{(1)}) m_j^{(2)}$$

where $M_j^{(2)} = \max_i p_{ij}^{(2)}$ and $m_j^{(2)} = \min_i p_{ij}^{(2)}$.

Thus

$$(q_{uj}^{(2)} - q_{vj}^{(2)}) \leq \sum^+ (q_{uk}^{(1)} - q_{vk}^{(1)}) (M_j^{(2)} - m_j^{(2)}) \leq (1 - N\delta_1)(1 - N\delta_2).$$

Thus $m_{ij}^{(k,l)} \leq M[\alpha^s + \alpha^t + \alpha^k + \alpha^l]$ where M is a constant.

$$\frac{1}{(n+1)^2} \sum_{k=0}^n \sum_{l=0}^n m_{ij}^{(k,l)} \leq \frac{M}{(n+1)^2} \sum_{k=0}^n \sum_{l=0}^n [\alpha^s + \alpha^t + \alpha^k + \alpha^l] \leq \frac{8M(n+1)}{(n+1)^2(1-\alpha)} = \frac{8M}{(n+1)(1-\alpha)}$$

$\rightarrow 0$ as $n \rightarrow \infty$.

Hence $p[|V_j(n) - \pi_j| > \varepsilon | X_0 = i] \rightarrow 0$ as $n \rightarrow \infty$.

Theorem 7.6

Assume we have an arbitrary sequence of Markov chains on a finite state space S with corresponding transition matrices $(A_n)_{n=1}^\infty$. Denote the entries of A_n by $[p_{ij}^{(n)}]_{i,j \in S}$. Assume that $\min_{i,j} p_{ij}^{(n)} = \delta_n \geq \varepsilon > 0$ for infinitely many n 's. Then $\lim_{n \rightarrow \infty} A_1 A_2 \dots A_n$ is Weak Ergodic. That is $\forall j \in S, \lim_{n \rightarrow \infty} |q_{ij}^{(n)} - q_{kj}^{(n)}| = 0, \forall i, k \in S$, where $[q_{ij}^{(n)}]_{i,j \in S}$ are the entries of $Q_n = A_1 A_2 \dots A_n$. That is, the effect of the initial state wears off.

Proof

Let $(A_n)_{n=1}^\infty$ be the transition matrices of this sequence.

Denote the transition matrix of the n -th chain by A_n and its entries by $[p_{ij}^{(n)}]_{i,j \in S}$.

Let $\lim_{n \rightarrow \infty} A_n = A$ and denote its entries by $[a_{ij}]_{i,j \in S}$.

Let δ be the minimum over all the entries of A .

Let N be the cardinality of S and $\delta_n = \min_{i,j} p_{ij}^{(n)}$.

Let $Q_n = A_1 A_2 \dots A_n$. and denote its entries by $[q_{ij}^{(n)}]_{i,j \in S}$.

Assume without loss of generality that A is with non-zero entries (otherwise there exists n_0 such that A^n is with non-zero entries for all $n \geq n_0$ since it is Ergodic).

the combination of this sequence. Then: $p[|V_j(n) - \pi_j| > \varepsilon] \rightarrow 0$ as $n \rightarrow \infty$, $\forall \varepsilon > 0$.

Proof

Let $X_0, X_1, \dots, X_n, \dots$ be the random variables which form these Markov chains (the values of them are in S).

Let $I_j(X_n) = 1$ if $X_n = j$; 0 otherwise. Then

$$V_j(n) = \frac{1}{n} \{I_j(X_1) + \dots + I_j(X_n)\}.$$

Let $q_{ij}^{(n)}$ be the n -th step probability of the combination.

Let i, j be two states in S .

We want to prove that $p[|V_j(n) - \pi_j| > \varepsilon] \rightarrow 0$ as $n \rightarrow \infty$, $\forall \varepsilon > 0$.

Notice that $EV_j(n) = \frac{1}{n+1} \sum_{m=0}^n q_{ij}^{(m)}$, which tends to π_j as n tends to ∞ .

By Chebyshev's Inequality:

$$p[|V_j(n) - \pi_j| > \varepsilon | X_0 = i] < \frac{E[(V_j(n) - \pi_j)^2 | X_0 = i]}{\varepsilon^2} \dots$$

Thus we have to prove that $E[(V_j(n) - \pi_j)^2 | X_0 = i] \rightarrow 0$ as $n \rightarrow \infty$.

$$E[(V_j(n) - \pi_j)^2 | X_0 = i] = \frac{1}{(n+1)^2} E\left[\sum_{k=0}^n (I_j(X_k) - \pi_j)^2 | X_i = 0\right]$$

$$= \frac{1}{(n+1)^2} \sum_{k=0}^n \sum_{l=0}^n m_{ij}^{(k,l)}$$

where

$$\begin{aligned} m_{ij}^{(k,l)} &= E[(I_j(X_k)I_j(X_l) | X_0 = i) - \pi_j E[I_j(X_k) | X_0 = i] - \pi_j E[I_j(X_l) | X_0 = i] + \pi_j^2] \\ &= q_{ij}^{(s)} q_{ij}^{(t)} - \pi_j q_{ij}^{(k)} - \pi_j q_{ij}^{(l)} + \pi_j^2 \end{aligned}$$

where $s = \min(k, l)$, $t = |k - l|$.

$q_{ij}^{(t)}$ stands for the probability of the combination from $s+1$ to $\max(k, l)$.

But we have $q_{ij}^{(n)} = \pi_j + \varepsilon_{ij}^{(n)}$, $|\varepsilon_{ij}^{(n)}| < C\alpha^n$.

Hence, by the principle of mathematical induction, (6) is valid for each natural number n .

Next, $\delta_n \rightarrow \delta$ and $\delta > 0 \Rightarrow \prod_{i=1}^n (1 - N\delta_i) \rightarrow 0$ as $n \rightarrow \infty$.

So if $R_j^n \rightarrow \pi_j$ then $r_j^n \rightarrow \pi_j$; indeed:

$$|q_{ij}^{(n)} - q_{kj}^{(n)}| \leq (R_j^{(n)} - r_j^{(n)}) < C\alpha^n$$

where C is constant and $0 \leq \alpha < 1, \forall i, j, k \in S$. That is

$$|q_{ij}^{(n)} - \pi_j| < C\alpha^n, \forall i, j \in S.$$

Notes:

- (1) If A has zero entries, then there exists n_0 such that A^n is with non-zero entries for each $n \geq n_0$, so there exists l such that $A_1 A_2 \dots A_l$ is with non-zero entries, so we may consider $A_1 A_2 \dots A_{kl}$ and take the limit as $k \rightarrow \infty$.
- (2) The limit of the combination does not depend on n , that is the limit exists. If we have the same transition matrix in each step, then $R_j^{(n)}$ is non-increasing (with respect to n) and $r_j^{(n)}$ is non-decreasing (See for example [3]⁹). If we do not have the same transition matrix, then they are almost monotonic (since we have a convergent sequence of transition matrices).

Corollary 7.2

Assume we have a sequence of Markov chains on a finite state space S . Assume that the limit of this sequence is Ergodic. Let $V_j(n)$ be the average number of staying in state j . Let $(\pi_j)_{j \in S}$ be the stationary distribution of

⁹ Billingsly, 1986, p.128

For $n = 1$:

$$\max_{i,j} p_{ij}^{(1)} \leq (1 - (N - 1)\delta_1) \text{ and}$$

$$\min_{i,j} p_{ij}^{(1)} \leq \delta_1 \Rightarrow (\max_i p_{ij}^{(1)} - \min_i p_{ij}^{(1)}) \leq (1 - N\delta_1)$$

$$\text{and since } Q_1 = P_1 \text{ then } (\max_i p_{ij}^{(1)} - \min_i p_{ij}^{(1)}) \leq (1 - N\delta_1) .$$

...(3)

For $n = 2$:

$$(q_{uj}^{(2)} - q_{vj}^{(2)}) = \sum_{k \in S} (q_{uk}^{(1)} - q_{vk}^{(1)}) p_{kj}^{(2)} \leq \sum^+ (q_{uk}^{(1)} - q_{vk}^{(1)}) M_j^{(2)} + \sum^- (q_{uk}^{(1)} - q_{vk}^{(1)}) m_j^{(2)}$$

where $M_j^{(2)} = \max_i p_{ij}^{(2)}$ and $m_j^{(2)} = \min_i p_{ij}^{(2)}$.

Thus

$$(q_{uj}^{(2)} - q_{vj}^{(2)}) \leq \sum^+ (q_{uk}^{(1)} - q_{vk}^{(1)}) (M_j^{(2)} - m_j^{(2)}) \leq (1 - N\delta_1)(1 - N\delta_2) .$$

Thus $(R_j^{(2)} - r_j^{(2)}) \leq (1 - N\delta_1)(1 - N\delta_2)$ (4)

Where $R_j^{(2)} = \max_i q_{ij}^{(2)}$ and $r_j^{(2)} = \min_i q_{ij}^{(2)}$.

Next, assume that: $(R_j^{(n-1)} - r_j^{(n-1)}) \leq \prod_{i=1}^{n-1} (1 - N\delta_i)$ (5)

Where $R_j^{(n-1)} = \max_i q_{ij}^{(n-1)}$ and $r_j^{(n-1)} = \min_i q_{ij}^{(n-1)}$.

Now we want to prove that it is correct for n .

$$(q_{uj}^{(n)} - q_{vj}^{(n)}) = \sum_{k \in S} (q_{uk}^{(1)} - q_{vk}^{(1)}) b_{kj}^{(n-1)} \leq \sum^+ (q_{uk}^{(1)} - q_{vk}^{(1)}) B_j^{(n-1)} + \sum^- (q_{uk}^{(1)} - q_{vk}^{(1)}) b_j^{(n-1)}$$

$$\leq \sum^+ (q_{uk}^{(1)} - q_{vk}^{(1)}) (B_j^{(n-1)} - b_j^{(n-1)}) \leq (1 - N\delta_1)(B_j^{(n-1)} - b_j^{(n-1)})$$

where $[b_{ij}^{(n-1)}]_{i,j \in S}$ are the entries of $A_2 A_3 \dots A_n$ and

$$B_j^{(n-1)} = \max_i b_{ij}^{(n-1)} \text{ and } b_j^{(n-1)} = \min_i b_{ij}^{(n-1)} .$$

Thus $(q_{uj}^{(n)} - q_{vj}^{(n)}) \leq (1 - N\delta_1) \prod_{i=2}^n (1 - N\delta_i) = \prod_{i=1}^n (1 - N\delta_i)$.

(3),(4) and (5) $\Rightarrow (R_j^{(n)} - r_j^{(n)}) \leq \prod_{i=1}^n (1 - N\delta_i)$ (6)

Where $R_j^{(n)} = \max_i q_{ij}^{(n)}$ and $r_j^{(n)} = \min_i q_{ij}^{(n)}$.

Remark: This theorem is proved algebraically, see for example [15]⁸. In this paper we introduce an analytical proof of this theorem.

Proof

Let $(A_n)_{n=1}^\infty$ be the transition matrices of this sequence. Denote the transition matrix of the n-th chain by A_n and its entries by $[p_{ij}^{(n)}]_{i,j \in S}$.

Let $\lim_{n \rightarrow \infty} A_n = A$ and denote its entries by $[a_{ij}]_{i,j \in S}$.

Let δ be the minimum over all the entries of A

Let N be the cardinality of S and $\delta_n = \min_{i,j} p_{ij}^{(n)}$.

Let $Q_n = A_1 A_2 \dots A_n$ and denote its entries by $[q_{ij}^{(n)}]_{i,j \in S}$.

Assume without loss of generality that A is with non-zero entries (otherwise there exists n_0 such that A^n is with non-zero entries for all $n \geq n_0$ since it is Ergodic).

Assume without loss of generality that for all n , A_n is with non-zero entries (otherwise there exists k_0 such that A_n is with non-zero entries for all $n \geq k_0$).

Now for any stochastic matrix A with entries $[p_{ij}]_{i,j \in S}$ and minimum over all its entries δ , the following relations are valid:

Denote the summation over $j \in S$ satisfying $p_{uj} \geq p_{vj}$ by \sum^+ and the summation over $j \in S$ satisfying $p_{uj} < p_{vj}$ by \sum^- for arbitrary states u and v in S . Then $\sum^+(p_{uj} - p_{vj}) + \sum^-(p_{uj} - p_{vj}) = 1 - 1 = 0$ (1)

And since $\sum^+ p_{vj} + \sum^- p_{uj} \geq N\delta$, then :

$$\sum^+(p_{uj} - p_{vj}) = 1 - \sum^- p_{uj} - \sum^+ p_{vj} \leq (1 - N\delta). \quad \dots(2)$$

Next, we will use induction on n to prove that:

$$(\max_i q_{ij}^{(n)} - \min_i q_{ij}^{(n)}) \leq \prod_{i=1}^n (1 - N\delta_i).$$

⁸ Senata, 1981, p.68

This sequence tends to a Markov chain whose transition matrix is:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The limit of this sequence is not Ergodic, the second and third states are periodic with period 2, thus $\lim_n A^n$ does not exist. Moreover, $\lim_{n \rightarrow \infty} A_1 A_2 \dots A_n$ does not exist.

More Examples can be found in [12]⁷.

Remarks

(1) If we consider arbitrary transition matrices of arbitrary Markov chains and we consider combinations of these chains, then we have all the possibilities. We can find combinations of Ergodic chains which are Ergodic, other combinations which are not Ergodic. We can find combinations of Non-Ergodic chains which are Ergodic, other combinations which are not Ergodic. We can find combinations of Ergodic chains with Non-Ergodic ones which are Ergodic, other combinations which are not Ergodic. Thus, for such a case we can reach no conclusion about the limit of the combination.

(2) If we consider a convergent sequence of Markov chains and if the limit of the sequence is not Ergodic, then the limit of the combination may exist and Ergodic (example 6.2), may exist and not Ergodic (example 6.4), may not exist (example 6.5).

7. Convergence Theorems (Ergodic Theorems: Limit Theorems) For Non-Stationary Markov Chains

Theorem 7.5

Assume we have a finite state space. Assume we have a sequence of Markov chains such that the limit of this sequence is an Ergodic Markov chain. Then the limit of the combination of the elements of this sequence exists and Ergodic .

⁷ Mallak, 1996, p.6

sequence is Ergodic. The limit of the sequence has the transition matrix

$$A, \text{ where: } A = \begin{bmatrix} \frac{1}{2} & (\frac{1}{2})^2 & (\frac{1}{2})^3 & \dots \\ \frac{1}{2} & (\frac{1}{2})^2 & (\frac{1}{2})^3 & \dots \\ \frac{1}{2} & (\frac{1}{2})^2 & (\frac{1}{2})^3 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}.$$

Moreover, $A_1 A_2 = A_2$ and $A_1 A_2 \dots A_n = A_n$, thus $\lim_{n \rightarrow \infty} A_1 A_2 \dots A_n = \lim_{n \rightarrow \infty} A_n = A$ which is Ergodic.

Example 6.4:

Let $(A_n)_{n=1}^{\infty}$ be a sequence of transition matrices of Markov chains, where:

$$A_n = \begin{bmatrix} (\frac{1}{2})^n & (\frac{1}{2})^{n-1} & \dots & \frac{1}{2} & (\frac{1}{2})^{n+1} & \dots \\ (\frac{1}{2})^n & (\frac{1}{2})^{n-1} & \dots & \frac{1}{2} & (\frac{1}{2})^{n+1} & \dots \\ (\frac{1}{2})^n & (\frac{1}{2})^{n-1} & \dots & \frac{1}{2} & (\frac{1}{2})^{n+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ (\frac{1}{2})^n & (\frac{1}{2})^{n-1} & \dots & \frac{1}{2} & (\frac{1}{2})^{n+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

For each fixed n , the n -th chain is Ergodic while the limit of the sequence is not Ergodic (the sequence tends to the zero matrix). Moreover, $A_1 A_2 = A_2$ and $A_1 A_2 \dots A_n = A_n$, thus $\lim_{n \rightarrow \infty} A_1 A_2 \dots A_n = \lim_{n \rightarrow \infty} A_n$ which is not Ergodic.

Example 6.5:

Let $(A_n)_{n=2}^{\infty}$ be a sequence of transition matrices of Markov chains, where:

$$A_n = \begin{bmatrix} \frac{1}{2} - \frac{1}{n} & \frac{1}{2} + \frac{1}{n} & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Example 6.2:

Let $(A_n)_{n=1}^\infty$ be a sequence of transition matrices of Markov chains, where:

$$A_n = \begin{bmatrix} \frac{1}{n} & 1 - \frac{1}{n} \\ 1 - \frac{1}{n} & \frac{1}{n} \end{bmatrix} \quad \text{This sequence tends to a Markov chain whose}$$

$$\text{transition matrix is: } A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The limit of this sequence is not Ergodic, it is periodic with period 2, thus

$\lim_n A^n$ does not exist while $\lim_{n \rightarrow \infty} A_1 A_2 \dots A_n$ exists and Ergodic.

Indeed:

$$\lim_{n \rightarrow \infty} A_1 A_2 \dots A_n = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Example 6.3:

Let $(A_n)_{n=1}^\infty$ be a sequence of transition matrices of Markov chains, where:

$$A_n = \begin{bmatrix} \frac{1}{2} & (\frac{1}{2})^2 & \dots & (\frac{1}{2})^{n-1} & 0 & (\frac{1}{2})^n & \dots \\ \frac{1}{2} & (\frac{1}{2})^2 & \dots & (\frac{1}{2})^{n-1} & 0 & (\frac{1}{2})^n & \dots \\ \frac{1}{2} & (\frac{1}{2})^2 & \dots & (\frac{1}{2})^{n-1} & 0 & (\frac{1}{2})^n & \dots \\ \frac{1}{2} & (\frac{1}{2})^2 & \dots & (\frac{1}{2})^{n-1} & 0 & (\frac{1}{2})^n & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{2} & (\frac{1}{2})^2 & \dots & (\frac{1}{2})^{n-1} & 0 & (\frac{1}{2})^n & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

That is, in the n-th chain, the n-th state is isolated, it is not reached from any state. For each fixed n, the n-th chain is not Ergodic, while the limit of the

$$B = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & \dots & \dots \\ \frac{1}{9} & 0 & \frac{8}{9} & 0 & \dots & \dots \\ \frac{1}{16} & 0 & 0 & \frac{15}{16} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{(1+i)^2} & 0 & 0 & 0 & \frac{(i+1)^2-1}{(i+1)^2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

where i is the number of the row. A is a transition matrix of an Ergodic Markov chain (since the first column is bounded). B is a transition matrix of a Non-Ergodic Markov chain, since:

$\lim_{n \rightarrow \infty} \prod_{k=2}^n \frac{k^2-1}{k^2} = \frac{1}{2} \Rightarrow f_{11} < 1$, so the first state is transient. Since the chain is irreducible, then all states are transient.

Now any combination of A and B is irreducible, it is obvious since $a_{ij} > 0 \Leftrightarrow b_{ij} > 0, \forall i, j \in S$ and both A and B are irreducible.

Next, any combination of A and B such that using A infinitely often is non-null persistent; since we use A infinitely often, once we use $A, q_{i1}^{(n)} = \frac{1}{2} \Rightarrow \sum_{n=1}^{\infty} q_{i1}^{(n)} \geq \sum_{k=1}^{\infty} \frac{1}{2} = \infty$. So, the first state is non-null persistent. The combination is irreducible, so all states are non-null persistent.

Next, any combination of A and B such that using both A and B infinitely often is not Ergodic, indeed the limit does not exist. Let C_1 be the class of any combination such that in the n -th step we have A and C_2 be the class of any combination such that in the n -th step we have B , both C_1 and C_2 have probability $\frac{1}{2}$. Now, for $C_1, q_{i1}^{(n)} = \frac{1}{2}$, for $C_2, q_{i1}^{(n)} \leq \frac{1}{4}, \forall i \in S$. Hence the limit of any such a combination does not exist.

P_1 . In two steps, denote it by $q_{ij}^{(2)}$, then $q_{ij}^{(2)} = \sum_{k \in S} p_{ik}^{(1)} p_{kj}^{(2)}$, where $[p_{ij}^{(1)}]_{i,j \in S}$, denote it by P_1 , is the one step transition matrix of the first chain and $[p_{ij}^{(2)}]_{i,j \in S}$, denote it by P_2 , is the one step transition matrix of the second chain. And in general, for any positive integer n , the n -th step probability of the combination, $q_{ij}^{(n)} = \sum_{k \in S} q_{ik}^{(n-1)} p_{kj}^{(n)}$, where $q_{ij}^{(n-1)}$ is the $(n-1)$ -th step probability of the combination and $p_{ij}^{(n)}$ is the one step probability of the n -th chain, denote its matrix by P_n . In matrix form $Q_n = P_1 P_2 \dots P_n$.

We will use the same definition of the original case, the stationary case, for irreducible, reducible, periodic, aperiodic, transient, persistent, null persistent and Ergodic state (chain).

The main question will be about Ergodicity of such combinations; that is whether the limit of $q_{ij}^{(n)}$ exists or not and the effect of the initial state whether it wears off or not for large n .

6. Examples Of Non-Stationary Markov Chains

Example 6.1:

Let A and B be two transition matrices of two Markov chains, where:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots & \dots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \dots & \dots \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & \dots & \dots \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

Then $V_j(n) \rightarrow \pi_j$ *a.s* ; where *a.s* stands for almost sure.

For the proof see [17]⁶.

4. Classification of Irreducible, Aperiodic Markov Chains

For an irreducible, aperiodic Markov chain there exist three possibilities:

- The chain is transient, $\forall i, j \in S, \lim_{n \rightarrow \infty} p_{ij}^{(n)} = 0$ and $\sum_n p_{ii}^{(n)} < \infty$. If the state space is finite, then this case is impossible.
- The chain is persistent, there exists no stationary distribution. $\forall i, j \in S, \lim_{n \rightarrow \infty} p_{ij}^{(n)} = 0$ and $\sum_n p_{ii}^{(n)} = \infty, \mu_j = \infty$. This is the null persistent case, if the state space is finite this case is impossible.
- The chain is Ergodic, there exists a stationary distribution, the chain is non-null persistent, $\forall i, j \in S, \lim_{n \rightarrow \infty} p_{ij}^{(n)} = \pi_j > 0$ and $\mu_j = 1 / \pi_j, \forall j \in S$.

From the previous classification we conclude that an irreducible, aperiodic Markov chain is Ergodic if the state space is finite.

In the previous sections we tried to give a summary for stationary Markov chains which is necessary for our work. Actually the theory of Markov chains is very rich, more details about Markov chains can be found in [1]-[8], [11]-[17].

5. Introduction to Non-Stationary Markov Chains

Assume we have different Markov chains with different transition matrices we will consider combinations of the probabilities of these chains. In other words, to get the higher probabilities of these combinations, we will use different transition matrices.

So, in one step, if we denote the probability of starting from state i reach state j in one step by $q_{ij}^{(1)}$, then it is the same as the one step transition probability of the first chain, denote it by $p_{ij}^{(1)}$, and the transition matrix

⁶ Taylor and Karlin, 1994, p.157

Notice that since $\sum_{j \in S} p_{ij}^{(n)} = 1$, the first alternative in theorem 2 is impossible

if S is a finite set, that is a finite irreducible Markov chain is persistent. The proof of this theorem can be found in [3], [7] and [16]⁴.

3. Convergence Theorems (Ergodic Theorems: Limit Theorems) For Stationary Markov Chains

Theorem 3.4

Suppose of an irreducible, aperiodic Markov chain that there exists a stationary distribution, that is a solution of

$$\sum_{i \in S} \pi_i p_{ij} = \pi_j, \forall j \in S, n = 1, 2, \dots \text{ satisfying } \pi_i > 0 \text{ and}$$

$$\sum_{i \in S} \pi_i = 1, \forall i \in S, \text{ then the Markov chain is persistent and}$$

$\lim_{n \rightarrow \infty} p_{ij}^{(n)} = \pi_j, \forall i, j \in S$. If the state space is finite, then $|p_{ij}^{(n)} - \pi_j| < A\rho^n$, where A is a constant and $0 \leq \rho < 1$; that is we have exponential convergence.

Remark: The main point of the conclusion is that since $p_{ij}^{(n)}$ reaches π_j for large n , the effect of the initial state wears off, that is the chain is very stable.

The proof of this well-known theorem can be found in many books, for example see [1]-[8],[14]-[17]⁵.

By the law of large numbers we can conclude the following corollary:

Corollary 3.1

Let X_1, X_2, \dots be a sequence of random variables which forms an Ergodic Markov chain. Let $I_j(X_n) = 1$ if $X_n = j$; 0 otherwise and

$$V_j(n) = \frac{1}{n} \{I_j(X_1) + \dots + I_j(X_n)\}.$$

⁴ For example see Billingsly, 1986, p.115

⁵ For example see Chen, 1992, p.157

- A state $i \in S$ is called periodic if $\exists t > 1$ such that $p_{ii}^{(n)} = 0$ unless $n = rt$, otherwise it is called aperiodic.
- A Markov chain is called irreducible if n such that $p_{ii}^{(n)} > 0, \forall i, j \in S$, otherwise it is called reducible.
- A Markov chain is called Ergodic if all its states are persistent, aperiodic and non-null persistent states; there exists a stationary distribution which is a set of probabilities $(\pi_j)_{j \in S}$ satisfying

$$\sum_{i \in S} \pi_i p_{ij} = \pi_j.$$

Theorem 2.2

A state i is persistent if and only if $p_i(X_n = i, i.o) = 1$ and $\sum_n p_{ii}^{(n)} = \infty$. A state i is transient if and only if $p_i(X_n = i, i.o) = 0$ and $\sum_n p_{ii}^{(n)} < \infty$, where $i.o.$ stands for infinitely often.

The proof of this theorem can be found in [3], [4], [16] and [17]².

Lemma 2.2

By the zero one law, $p_i(X_n = i, i.o)$ is either 0 or 1.

For the proof of this lemma see [3]³.

Theorem 2.3

If a Markov chain is irreducible, then either all states are transient, $p_i(\cup_j (X_n = j, i.o)) = 0, \forall i, j \in S$ and $\sum_n p_{ii}^{(n)} < \infty$. Or all states are persistent,

$$p_i(\cap_j (X_n = j, i.o)) = 1, \forall i, j \in S \text{ and } \sum_n p_{ii}^{(n)} = \infty.$$

² For example see Taylor & Karlin, 1994, p.207

³ Billingsly, 1986, p.114

Thus a Markov chain is stationary if it has the same transition matrix in each step, otherwise it is non-stationary.

Theorem 1.1: An Existence Theorem

Suppose that $P = [p_{ij}]_{i,j \in S}$ is a stochastic matrix and that π_i 's are nonnegative numbers satisfying $\sum_{i \in S} \pi_i = 1$. Then there exists on some probability space a Markov chain X_0, X_1, X_2, \dots with initial probability π_i and transition probability p_{ij} . For the proof of this theorem see [3]¹.

Although strictly speaking the Markov chain is the sequence X_0, X_1, X_2, \dots , by this theorem one can say the chain where the matrix P together with the initial probability π_i or even P with some unspecified set of π_i .

2. Classification of Markov Chains

Let $f_{ii}^{(n)} := p_i(X_1 \neq i, X_2 \neq i, \dots, X_{n-1} \neq i, X_n = i)$, it means the probability of the first visit to the state i at time n . Let

$f_{ii} := \sum_{n=1}^{\infty} f_{ii}^{(n)} = p_i(\bigcup_{i=1}^{\infty} (X_n = i))$, it means the probability of visiting

the state i infinitely often. Let $\mu_i := \sum_{n=1}^{\infty} n f_{ii}^{(n)}$ it means the expectation of visiting the state i infinitely often, it is called the mean recurrence time.

Definition 2.5

- A state $i \in S$ is called persistent if $f_{ii} = 1$, transient if $f_{ii} < 1$.
- A state $i \in S$ is called null persistent if the mean recurrence time $\mu_i = \infty$.

¹ Billingsly, 1986, p.112

what Markov property requires. The initial probabilities are $\pi_i = P[X_0 = i]$, the π_i 's are nonnegative and add to 1.

Definition 1.2

A square matrix P is called a stochastic matrix if all its entries are non-negative and the summation of the elements of each row is 1. It is easy to see that stochastic matrices are closed under multiplication.

Lemma 1.1

The product of two stochastic matrices is again a stochastic matrix.

Proof

Let $A = [a_{ij}]_{i,j \in S}$ and $B = [b_{ij}]_{i,j \in S}$ be two stochastic matrices.

$$\text{Then } \sum_{j \in S} \sum_{k \in S} a_{ik} b_{kj} = \sum_{k \in S} a_{ik} \sum_{j \in S} b_{kj} = (1)(1) = 1.$$

In particular, if P is a stochastic matrix, then $P^2, P^3, \dots, P^n, \dots$ are stochastic matrices.

Definition 1.3

Let $P = [p_{ij}]_{i,j \in S}$ be a stochastic matrix, then P is called the one step transition (probability) matrix of this Markov chain. p_{ij} means the probability of starting from the state i reaching the state j (in one step). $P^2 = [p_{ij}^{(2)}]_{i,j \in S}$ is the second step transition matrix. $p_{ij}^{(2)}$ means the probability of starting from the state i reaching the state j in two steps. For any positive integer n , $P^n = [p_{ij}^{(n)}]_{i,j \in S}$ is the n -th step transition matrix. $p_{ij}^{(n)}$ means the probability of starting from the state i reaching the state j in n steps.

Definition 1.4

A sequence of random variables $(X_n)_{n \geq 1}$ is called a stationary sequence (homogenous or shift invariant) if for each natural numbers k and n , (X_1, X_2, \dots, X_n) and $(X_{k+1}, X_{k+2}, \dots, X_{k+n})$ have the same distribution..

1. Introduction

Markov chains are stochastic processes which are ways of quantifying the dynamic relationships of sequences of random variables. Stochastic models play an important role in many areas of the natural and engineering sciences [1]-[18]. Indeed if we have a sequence of random variables with values in a discrete set, a countable set, then any such a sequence can form a Markov chain, which is conditional probabilities relating the elements of this sequence.

The most interesting object of the theory of Markov chains is the asymptotic behavior of these probabilities. The most interesting case when we have independence of the initial state; that is starting from any state, the particle reaches the desired state almost with the same probability. A Markov chain satisfying this is called Ergodic. We may characterize Ergodic Markov chains by the saying: "All ways lead to Rome".

Next, we introduce a mathematical introduction.

Definition 1.1

Let S be a discrete set, finite or countably infinite. Suppose to each pair $i, j \in S$ there is assigned a nonnegative number p_{ij} such that these numbers satisfy the constraint $\sum_{j \in S} p_{ij} = 1, \forall i \in S$. Let $X_0, X_1, \dots, X_n, \dots$ be a sequence of random variables whose ranges are contained in S . The sequence is a Markov chain if:

$$P[X_{n+1} = j | X_0 = i_0, \dots, X_n = i_n] = P[X_{n+1} = j | X_n = i_n] = p_{i_n j}, \forall n$$

and every sequence $\{i_0, i_1, \dots, i_n\} \subseteq S$ for which

$P[X_0 = i_0, \dots, X_n = i_n] > 0$, this property is called Markov property. S is called the state space or the phase space of the Markov chain. The elements of S are thought of as the possible states of a system, X_n representing the state at time n . The sequence or process X_0, X_1, X_2, \dots then represents the history of the system, which evolves in accordance with the probability law defined above. The conditional distribution of the next state X_{n+1} given the present state X_n must not further depend on the past X_0, \dots, X_{n-1} . This is

ملخص

نظريات في نهايات سلاسل ماركوف المعدودة غير المتجانسة
١- نقوم في هذا البحث بدراسة نهايات سلاسل ماركوف المعدودة غير المتجانسة . حيث
أثبتنا أنه اذا كان هناك متتالية من سلاسل ماركوف الاركوديكية فان تركيب من هذه العناصر
يكون اركوديكي اذا كانت مجموعة الحالات منتهية ويكون التركيب كذلك في حالة ان تكون
مجموعة الحالات غير منتهية ولكن تحت شرط اضافي . كما أثبتنا أن نهاية تركيب من متتالية
عشوائية تكون اركوديكي ضعيف تحت شرط معين , وتحت نفس الشرط تكون نهاية تركيب
من متتالية من سلاسل ماركوف الاستوكاستيكية المزدوجة هي اركوديكية
٢- هذا البحث مرتب بالطريقة التالية :-

- الجزء (١)- الجزء (٤) :- نظرة عامة عن سلاسل ماركوف وتصنيفاتها
- الجزء (٥) :- مقدمة عن سلاسل ماركوف غير المتجانسة .
- الجزء (٦) :- أمثلة عن سلاسل ماركوف غير المتجانسة .
- الجزء (٧) :- نظريات حول سلاسل ماركوف غير المتجانسة .
- الجزء (٨) :- ملاحظات حول الموضوع .
- ٣- مصطلحات أساسية :

سلاسل ماركوف , استوكاستيكي, استوكاستيكي مزدوج, دوري , غير دوري , اصراري ,
غير اصراري , مصفوفة الانتقال , نظرية اركوديكي .

Abstract

In this paper we study the Ergodicity of non-stationary discrete time Markov chains. We prove that given a sequence of Ergodic Markov chains, then the limit of the combination of the elements of this sequence is again Ergodic (under additional condition if the state space is infinite). We also prove that the limit of an arbitrary sequence of Markov chains is weak Ergodic if it satisfies some condition. Under the same condition, the limit of the combination of doubly stochastic sequence of Markov chains is Ergodic.

Keywords: Markov Chain, Stochastic, Doubly Stochastic, Irreducible, Aperiodic, Persistent, Transient, Ergodic, Transition Matrix, Ergodic Theorem.

The paper is organized in the following way. In the first four sections we give a general review of the theory of Markov chains: definitions, classifications of the chains and main theorems. In section 5 we introduce the concept of non-stationary Markov chains. In section 6 we give some examples of non-stationary Markov chains. In section 7 we give some limit theorems for non-stationary Markov chains which is our main result. In section 8 we give some remarks.

***ERGODIC (LIMIT) THEOREMS FOR NON-STATIONARY
DISCRETE TIME MARKOV CHAINS***

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