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This equation can be written

$$\sum_{i,j,j',k,k',l} (T_i T_k \frac{\partial x^l}{\partial p^{k'}_{j'}} x^j + T_i \frac{\partial T_k}{\partial p^{k'}_{j'}} x^j x^l) dp^i_j \wedge dp^k_l \wedge dp^{k'}_{j'} = 0 \tag{21}$$

This equality is satisfied if and only if the coefficients satisfy the following symmetry conditions:

$$T_i x^j (T_k \frac{\partial x^l}{\partial p^{k'}_{j'}} - T_{k'} \frac{\partial x^{j'}}{\partial p^k_l}) + T_{k'} x^{j'} (T_i \frac{\partial x^j}{\partial p^k_l} - T_k \frac{\partial x^l}{\partial p^i_j}) + T_k x^l (T_{k'} \frac{\partial x^{j'}}{\partial p^i_j}$$

$$-T_i \frac{\partial x^j}{\partial p^{k'}_{j'}}) + (T_i \frac{\partial T_k}{\partial p^{k'}_{j'}} - T_k \frac{\partial T_i}{\partial p^{k'}_{j'}}) x^j x^l + (T_{k'} \frac{\partial T_i}{\partial p^k_l} - T_i \frac{\partial T_{k'}}{\partial p^k_l}) x^j x^{j'}$$

$$+(T_k \frac{\partial T_{k'}}{\partial p^i_j} - T_{k'} \frac{\partial T_k}{\partial p^i_j}) x^{j'} x^l = 0$$

Using the fact that $T_i/T_k = \lambda_i \lambda_k$ and dividing by $T_i T_k$, we get the desired conditions. This concludes the proof. ∎

We can check easily that conditions (18) imply both (a) and (b) in theorem 1. If we take $i = k = k'$ then (18) is equivalent to (a) and when $i = k$ then (18) is nothing but (b). Moreover, the conditions we obtained above imply the Slutsky conditions in the single constraint case and the conditions given in Aloqeili [1] using the homogeneity properties of x and the budget constraint(s) $Px = y$.

5 Possible applications of the results

In an important article published in the late nineties, Browning and Chiappori [3] tested the individual demand function model econometrically. For the first time they got positive results regarding the validity of the Slutsky relations. In their article, they used data collected in different provinces of Canada. Browning and Chiappori have shown that Slutsky relations, zero-homogeneity and Walras law fully characterize the individual demand function. Moreover, they setup a model for household and validated it econometrically.

In a similar way, the results we got here can be tested econometrically. The demand functions depend on prices and income which are known variables and individual demands are observable. The problem in this kind of econometric test is that we cannot put the people in laboratory conditions. Indeed, we need to observe their normal behavior in the market. The solution to this problem is to consider a large country such as Canada in which prices change between provinces because of different tax regimes. This is indeed the procedure that was adopted by Browning and Chiappori.

Theorem 2 *Let $x(P) \in \mathbb{R}^n$ be a function of class C^2 in a neighborhood $\mathcal{U}$ of some point $\bar{P} \in \mathbb{R}^{mn}_{++}$ and satisfying $Px = 0$ and homogeneous of degree zero in p^i for all $i \leq m$. Then there exist some functions V and m positive functions $\lambda_1, ..., \lambda_m$ defined on some neighborhood $\mathcal{V} \subset \mathcal{U}$ such that*

$$\frac{\partial V}{\partial p^i_j} = -\lambda_i x^j$$

if and only if there exists a family of positive functions λ_{ik}, satisfying condition $\mathcal{H}$, defined on $\mathcal{V}$ such that

$$T_k^{jj'll'} = \lambda_{ki} T_i^{lj'jl'} \tag{17}$$

$$\frac{\partial x^l}{\partial p^{k'}_{j'}} x^j - \frac{\partial x^j}{\partial p^{k'}_{j'}} x^l + \lambda_{k'k}\left(\frac{\partial x^j}{\partial p^k_l} x^{j'} - \frac{\partial x^{j'}}{\partial p^k_l} x^j\right) + \lambda_{k'i}\left(\frac{\partial x^{j'}}{\partial p^i_j} x^l - \frac{\partial x^l}{\partial p^i_j} x^{j'}\right)$$
$$+\lambda_{ik}\left(\frac{\partial \lambda_{ki}}{\partial p^{k'}_{j'}} x^l - \frac{\partial \lambda_{k'i}}{\partial p^k_l} x^{j'}\right) x^j + \lambda_{ki} \frac{\partial \lambda_{k'k}}{\partial p^i_j} x^{j'} x^l = 0 \tag{18}$$

where $\lambda_{ik} = \frac{\lambda_i}{\lambda_k}$. Moreover, V is quasiconvex with respect to p^i for all i if and only if the restriction of $D_{p^i} x$ on $\{x\}^\perp$ is negative semidefinite.

Proof: The first conditions are the necessary ones. It suffices to show that the second ones are equivalent to

$$\Omega^{jj'll'} \wedge d\Omega^{jj'll'} = 0$$

To simplify notations, we omit the superscripts of T_i and T_k. In fact, since the quotient in (16) doesn't depend on j, j', l, l' then the calculation is the same for any j, j', l, l'. We can also forget $\frac{1}{T_1}$ because the exterior products are invariant for such changes. We thus need to find the conditions such that

$$\Omega \wedge d\Omega = 0 \tag{19}$$

where

$$\Omega = \sum_i T_i \omega^i$$

and

$$d\Omega = \sum_k T_k d\omega^k + \sum_k dT_k \wedge \omega^k$$

Therefore, $\Omega \wedge d\Omega = 0$ takes the form

$$\sum_{i,k=1}^{m} T_i T_k \omega^i \wedge d\omega^k + \sum_{i,k=1}^{m} T_i dT_k \wedge \omega^k \wedge \omega^i = 0 \tag{20}$$

Notice that the tensor T is symmetric with respect to the indices jl. We get the following relation between λ_i and λ_k for any i and k

$$\lambda_i = \frac{T_i^{lj'jl'}}{T_k^{jj'll'}} \lambda_k \tag{10}$$

Set $k = k_0$ for some $k_0 \in \{1, ..., n\}$, we can express $\lambda_i \ \ \forall i$ as a multiple of λ_{k_0}.

$$\lambda_i = \frac{T_i^{lj'jl'}}{T_{k_0}^{jj'll'}} \lambda_{k_0} \tag{11}$$

Take $k_0 = 1$. The decomposition $dV = -\sum_i \lambda_i \omega^i$ then writes down

$$dV = -\frac{\lambda_1}{T_1^{jj'll'}} \sum_i T_i^{lj'jl'} \omega^i \quad \forall 1 \leq j, j', l, l' \leq n \tag{12}$$

We define, $\forall 1 \leq j, j', l, l' \leq n$, the 1-form

$$\Omega^{jj'll'} = \frac{1}{T_1^{jj'll'}} \sum_i T_i^{lj'jl'} \omega^i \tag{13}$$

Our problem now boils down to finding the necessary and sufficient conditions such that

$$-\frac{1}{\lambda_1} dV = \frac{1}{T_1^{jj'll'}} \sum_i T_i^{lj'jl'} \omega^i \quad \forall 1 \leq j, j', l, l' \leq n \tag{14}$$

Which is equivalent to

$$\frac{1}{\lambda_1} dV = \Omega^{jj'll'} \quad \forall 1 \leq j, j', l, l' \leq n \tag{15}$$

Notice that since

$$\frac{T_i^{jj'll'}}{T_k^{lj'jl'}} = \frac{\lambda_i}{\lambda_k} := \lambda_{ik} \tag{16}$$

Then the quotient on the left doesn't depend on j, j', l, l'. Using Frobinus' theorem, it follows that

$$\frac{1}{\lambda_1} dV = \Omega^{jj'll'} \text{ if and only if } \Omega^{jj'll'} \wedge d\Omega^{jj'll'} = 0$$

The functions λ_{ik} defined above have the following homogeneity properties:
Condition $\mathcal{H}$:
The function λ_{ik}, for any $i \neq k$, is homogeneous of degree -1 in p^i, of degree 1 in p^k and of degree zero in $p^{k'}$, for all $k' \neq i$ and $k' \neq k$.
Now we are able to give the necessary and sufficient conditions for mathematical integration.

Taking the exterior derivative and multiplying by $\omega^1 \wedge ... \wedge \omega^m$, we get

$$\sum_{i=1}^{m} \lambda_i d\omega^i \wedge \omega^1 \wedge ... \wedge \omega^m = 0 \tag{6}$$

Writing $d\omega^i$ explicitly and substituting for $\omega^1, ..., \omega^m$, the above equation writes down

$$\sum_{i,j,k,l} \left(\lambda_i \frac{\partial x^j}{\partial p_l^k} - \lambda_k \frac{\partial x^l}{\partial p_j^i} \right) dp_l^k \wedge dp_j^i \wedge \sum_{i_1,...,i_m} x^{i_1}...x^{i_m} dp_{i_1}^1 \wedge ... \wedge dp_{i_m}^m = 0 \tag{7}$$

Rewrite this equation as follows:

$$\sum_{i,j,j',l} \lambda_i \left(\frac{\partial x^j}{\partial p_l^i} - \frac{\partial x^l}{\partial p_j^i} \right) x^{j'} dp_l^i \wedge dp_j^i \wedge dp_{j'}^i \wedge \Gamma^i$$

$$+ \sum_{i,j,k,l,j',l'} \left(\lambda_i \frac{\partial x^j}{\partial p_l^k} - \lambda_k \frac{\partial x^l}{\partial p_j^i} \right) x^{j'} x^{l'} dp_j^i \wedge dp_{j'}^i \wedge dp_l^k \wedge dp_{l'}^k \wedge \Delta^{ik} = 0 \tag{8}$$

where Γ^i is the $(m-1)$-form defined by

$$\Gamma^i = \pm \omega^1 \wedge ... \wedge \widehat{\omega^i} \wedge ... \wedge \omega^m$$

and Δ^{ik} is the $(m-2)$-form given by

$$\Delta^{ik} = \pm \omega^1 \wedge ... \wedge \widehat{\omega^i} \wedge .... \wedge \widehat{\omega^k} \wedge ... \wedge \omega^m$$

where the hat means that the corresponding 1-form is not included in the product. The first summation gives conditions (a), while the second one gives conditions (b). ∎

The signs in the formulas of Γ^i and Δ^{ik} come from the permutations applied on the product $\omega^1 \wedge ... \wedge \omega^m$ in order to get equation (8). The conditions given in theorem1 are necessary but not sufficient. In the next section we give the necessary and sufficient conditions for both mathematical and economic integration.

4.1 The sufficient conditions

Now, we want to find the sufficient conditions for mathematical integration. The necessary conditions enable us to get relations between the functions $\lambda_1, ..., \lambda_m$. Define the tensor $T_k^{jj'll'}$ such that (b) writes down

$$\lambda_i T_k^{jj'll'} = \lambda_k T_i^{lj'jl'} \tag{9}$$

We define a family of 1-forms

$$\omega^i = \sum_j x^j dp^i_j \tag{4}$$

Therefore, relations (3) and (4) imply that

$$dV = -\sum_i \lambda_i \omega^i \tag{5}$$

Our problem now can be formulated as follows: What are the necessary and sufficient conditions for the existence of $m+1$ functions $V, \lambda_1, ..., \lambda_m$ satisfying (5). This is the mathematical integration problem in our setting. One can easily get some necessary conditions. Using the first order conditions $D_{p^i}V = -\lambda_i x$, we conclude that the following conditions hold, for all i and k, on the subspace $\{x\}^\perp$:

(i) The $n \times n$ matrix $D_{p^i}x$ is symmetric and negative semidefinite.

(ii) For any i and k, the matrices $D_{p^i}x$ and $D_{p^k}x$ are proportional; that is, $\lambda_i(D_{p^k}x) = \lambda_k(D_{p^i}x)$.

The following theorem rewrites conditions (i) and (ii) in a more explicit way:

Theorem 1 *Let $x(P)$ be the solution of problem $(\mathcal{P})$ and $\lambda = (\lambda_1, ..., \lambda_m)$ be the corresponding Lagrange multipliers, then x and λ satisfy the following necessary conditions:*
(a) $\forall i = 1, ..., m$

$$x^k\left(\frac{\partial x^j}{\partial p^i_l} - \frac{\partial x^l}{\partial p^i_j}\right) + x^l\left(\frac{\partial x^k}{\partial p^i_j} - \frac{\partial x^j}{\partial p^i_k}\right) + x^j\left(\frac{\partial x^l}{\partial p^i_k} - \frac{\partial x^k}{\partial p^i_l}\right) = 0 \quad \forall j, k, l.$$

(b) $\forall 1 \le i, k \le m,\ 1 \le j, j', l, l' \le n.$

$$\lambda_i\left(\frac{\partial x^j}{\partial p^k_l}x^{j'}x^{l'} - \frac{\partial x^j}{\partial p^k_{l'}}x^{j'}x^l + \frac{\partial x^{j'}}{\partial p^k_{l'}}x^j x^l - \frac{\partial x^{j'}}{\partial p^k_l}x^j x^{l'}\right) =$$

$$\lambda_k\left(\frac{\partial x^l}{\partial p^i_j}x^{j'}x^{l'} - \frac{\partial x^{l'}}{\partial p^i_j}x^{j'}x^l + \frac{\partial x^{l'}}{\partial p^i_{j'}}x^j x^l - \frac{\partial x^l}{\partial p^i_{j'}}x^j x^{l'}\right)$$

Proof: Let x be the solution of problem $(\mathcal{P})$. So we can define the 1-forms $\omega^1, ..., \omega^m$ as in (4). Let V be the value function of problem $(\mathcal{P})$, then

$$dV = -\sum_{i=1}^m \lambda_i \omega^i$$

It follows that the above equality is satisfied if and only if

$$x^i\left(\frac{\partial x^j}{\partial p_k}-\frac{\partial x^k}{\partial p_j}\right)+x^k\left(\frac{\partial x^i}{\partial p_j}-\frac{\partial x^j}{\partial p_i}\right)+x^j\left(\frac{\partial x^k}{\partial p_i}-\frac{\partial x^i}{\partial p_k}\right)=0 \qquad (2)$$

These are the necessary and sufficient conditions for *mathematical integration.* This means that for given C^2 function $x(p)$, there exist two functions V and λ satisfying (1) if and only if x satisfies (2). Notice that, from equation (1), the restriction of $D_p x$ to $\{x\}^\perp$ is symmetric and positive semidefinite since V is quasiconvex. This is exactly the meaning of conditions (2).
It is important to notice that the conditions given in (2) are necessary and sufficient for the existence of V and λ such that $D_pV = -\lambda x$. On the other hand, in order for V to be quasiconvex and $\lambda > 0$, it is necessary and sufficient that the restriction of $D_p x$ to $\{x\}^\perp$ be negative semidefinite using the convex Darboux theorem, [8].
In the next section we will always assume, to simplify notation, that $1 \leq i, k, k' \leq m$ and $1 \leq j, j', l, l' \leq n$.

4 The multi-constraint case

Now, we consider the following problem

$$(\mathcal{P})\begin{cases} \max\ U(x) \\ Px \leq 0 \end{cases}$$

Our goal is to find the conditions satisfied by the solution to this problem. We suppose that all entries of the $m\times n$ matrix are strictly positive and that $m < n$. We introduce, as we did above, the value function of this problem

$$V(P) = \max\{U(x) - \sum_{i=1}^{m}\lambda_i\sum_{j=1}^{n}p^i_j x^j\}$$

One can easily show that V is quasiconvex and homogeneous of degree zero with respect to p^i, where p^i is the ith row of the matrix P.
Let $x(P)$ be the solution of problem $(\mathcal{P})$ and $\lambda(P) = (\lambda_1(P), ..., \lambda_m(P)) \in \mathbb{R}^m_{++}$ be the associated Lagrange multipliers corresponding to the m linear constraints. We assume that the solution $x(P)$ satisfies the equality constraints $Px(P) = 0$. It is homogeneous of degree zero with respect to p^i, for all i. Moreover, for any i, λ_i is homogeneous of degree -1 with respect to p^i and homogeneous of degree zero with respect to p^k for all $k \neq i$. Differentiating V with respect to p^i_j, we get

$$\frac{\partial V}{\partial p^i_j} = -\lambda_i x^j \qquad (3)$$

3 The single constraint case

We study firstly the single constraint case. So, we consider the problem

$$\max\ U(x)$$
$$p.x \leq 0$$

where , $p \in \mathbb{R}^n_{++}$ is the vector of prices and U is the utility function that is of class C^3 and strongly quasiconcave[2]. Introduce the value function, or in economic terms the indirect utility function

$$V(p) = \max\{U(x) - \lambda p.x\}$$

where λ is the Lagrange multiplier. The assumptions on U implies, using the implicit function theorem, that the solution to this problem $x(p)$ and the associated Lagrange multiplier $\lambda(p)$ are of class C^2. We assume that the solution $x(p) \in \mathbb{R}^n$ satisfies $p.x(p) = 0$ and that $\lambda > 0$. Notice that both of $V(p)$ and $x(p)$ are homogeneous of degree zero. Deriving V with respect to p, we get

$$\frac{\partial V}{\partial p_i} = -\lambda x^i \quad i = 1, ..., n \tag{1}$$

Define the the differential 1-form

$$\omega = \sum_i x^i dp_i$$

Equation (1) implies that

$$\frac{-1}{\lambda} dV(p) = \omega$$

This means that the 1-form ω is proportional to the differential of some function. It follows that

$$\omega \wedge d\omega = 0$$

where $d\omega$ is the exterior derivative of ω and is given by

$$d\omega = \sum_{j,k} \frac{\partial x^k}{\partial p_j} dp_j \wedge dp_k$$

and $\wedge$ is the exterior (wedge) product. Then

$$\omega \wedge d\omega = \sum_{i,j,k} x^i \frac{\partial x^k}{\partial p_j} dp_i \wedge dp_j \wedge dp_k = 0$$

[2] The Hessian matrix of U is negative definite on the subspace orthogonal to its gradient at each point.

can rationalized by means of some direct utility function U. This problem was addressed in [5] by Chiappori and Ekeland in the single constraint case. For the multi-constraint case, as in Aloqeili [1], it is impossible to retrieve a quasiconcave direct utility function. However, we can define a class of functions that is stable under duality, see Epstein [4]. However, we focus our attention here on the necessary conditions.
All our analysis and results hold locally since we mainly depend on the local version of Frobenius' theorem[1]. So, all functions are defined in a sufficiently small neighborhood of some given point $\bar{P}$.
In the next section we give some economic problems that motivates our research. We then discuss the single constraint problem and finally we consider the multi-constraint case.

2 Economic Motivation

Suppose that a consumer has a utility function U, and faces prices $p \in \mathbb{R}^n_{++}$. His problem is to choose the consumption (normal) bundle that maximizes his utility among those which are affordable. We suppose that his income is the market value of an initial endowment $\omega \in \mathbb{R}^n_+$. Mathematically, the consumer objective is to solve the following optimization problem:

$$\begin{gathered} \max \ U(x) \\ p.x \leq p.\omega \end{gathered}$$

Let $z = x - \omega$ be the individual excess demand where x is the actual demand function. Then the above problem writes down:

$$\begin{gathered} \max \ U(z+\omega) \\ p.z \leq 0 \end{gathered}$$

This problem can be extended easily to the multi-constraint case. Let P be an $m \times n$ matrix whose rows represent the market prices prevailing in m possible states of nature. Suppose that the consumer has a vector of endowments ω that is, of course, independent of the states of nature (he has ω now before the realization of any of the states of nature). The consumer's problem, in this case, writes down:

$$\begin{gathered} \max \ U(x) \\ Px \leq P\omega \end{gathered}$$

Rewriting this problem using excess demand function, we get

$$\begin{gathered} \max \ U(z+\omega) \\ Pz \leq 0 \end{gathered}$$

We see that, in the two previous examples, the economic problems fall in the same category as mentioned above.

[1] Frobenius' theorem states that $d\omega(p) = 0$ in a neighbourhood of some point $\bar{p}$ if and only if there is a function f such that $\omega = f$ in the same neighbourhood.

On the characterization of $\arg\max\{U(x) \mid Px = 0\}$

Marwan Aloqeili*

Abstract

In this paper, we use exterior differential calculus notations to get the conditions that characterize the solution to the utility maximization problem subject to the constraints $Px = 0_{\mathbb{R}^m}$ where P is an $m \times n$ matrix.

Key Words: Direct utility, Indirect utility, Differential forms, Slutsky matrix.

1 Introduction

In this paper we focus our attention on the application of exterior differential calculus tools to some optimization problems arising from economics. We will use the notions of exterior differential calculus to "characterize" the solutions of two types of maximization problems: single constraint and multi-constraint problems. These notions provide very efficient tools to solve such problems. Hurwicz [5] and [9] was the first who applied differential geometric tools to economics problems. Then Chiappori and Ekeland [6] and [7] used extensively these notions to solve many problems in economics. We will see how efficient are these tools. Moreover, they form an appropriate framework since the results are invariant under a change of coordinates.

In connection with this article, Aloqeili [1] and [2] solved the problem of characterizing the solution of multi-constraint problems in which the constraints take the form $Px = y$, where P is an $m \times n$ matrix and x is a vector in $\mathbb{R}^n_{++}$ represents a commodity bundle and $y \in \mathbb{R}^m_{++}$ represents income in many states of nature.

In this article, the constraints have the form $Px = 0_{\mathbb{R}^m}$. Such problems arise in general equilibrium theory and more precisely in the theory of excess demand functions and their characterization.

We will not treat the characterization problem in its general form. In particular, we don't address the problem of of weather a certain excess demand function

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On the characterization of $\arg\max\{U(x) \mid Px = 0\}$

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